STRUCTURAL FORMULAE IN LOGARITHMIC GAUGED LINEAR SIGMA MODELS I: THE TROPICAL DECOMPOSITION FORMULA

QILE CHEN, FELIX JANDA, AND YONGBIN RUAN

Abstract. We prove a tropical decomposition formula for the reduced virtual cycle in log GLSM. This allows to express Gromov–Witten invariants of complete intersections or Fan–Jarvis–Ruan–Witten invariants in terms of two ingredients: (1) integrals against the canonical virtual cycle of log R-maps and (2) effective invariants.

Contents

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1. INTRODUCTION

1.1. **Background on logarithmic Gauged Linear Sigma Models.** This and the subsequent works [19, 17] form the geometric foundation for computations using logarithmic Gauged Linear Sigma Models (log GLSM), as introduced in [18] using the logarithmic machinery of [2, 3].

A key concept from [18] is the notion of *log R-maps*. Consider the torus $\mathbb{C}^*_\omega \cong \mathbb{G}_m$ and the stack $\mathbf{BC}^*_\omega = [\operatorname{Spec}\mathbf{k}/\mathbb{C}^*_\omega]$ parameterizing line bundles. Let $\mathcal{L}_\omega = [\mathbb{C}_\omega/\mathbb{C}^*_\omega]$ be the universal line bundle over $\mathbf{BC}^*_\omega$, where $\mathbb{C}_\omega$ is the $\mathbb{C}^*_\omega$-representation of weight 1. A *log R-map* over a log scheme S is a (2-)commutative triangle

(1.1)
$$\begin{tikzcd} & \mathfrak{P} \arrow[d] \\ C \arrow[ur, "f"] \arrow[r, "\omega^{\log}_{C/S}"'] & \mathbf{BC}^*_\omega \end{tikzcd}$$

where $\mathfrak{P} \to \mathbf{BC}^*_\omega$ is a proper, DM-type morphism of log stacks (the symbol $\mathfrak{P}$ is the fraktur letter "P"), and $C \to S$ is a log curve, the bottom arrow is induced by the log cotangent bundle $\omega^{\log}_{C/S}$, and f is a log map. Removing all log structures from (1.1), defines *underlying R-map* over the underlying scheme $\underline{S}$.

To allow applications to both Gromov–Witten theory and Fan–Jarvis–Ruan–Witten (FJRW) theory, we consider the hybrid targets $\mathfrak{P} \to \mathbf{BC}^*_\omega$ as in introduced in [18] and reviewed in §2.1.1. Roughly speaking, $\mathfrak{P}$ is constructed from a smooth Deligne–Mumford stack $\mathcal{X}$ with projective coarse moduli by taking a weighted projective compactification of a vector bundle twisted by powers of $\omega^{\log}_C$. Indeed the log structure $\mathcal{M}_{\mathfrak{P}}$ over $\mathfrak{P}$ is the divisorial log structure defined by a smooth divisor $\infty \subset \mathfrak{P}$ such that $\mathfrak{P}^\circ := \mathfrak{P} \setminus \infty$ is the vector bundle (2.4). Denote by $0_{\mathfrak{P}} \subset \mathfrak{P}^\circ$ its zero section. Furthermore, there is a canonical projection $\mathfrak{P} \to \mathcal{X}$ as in (2.6), which on the Gromov–Witten side is crucial for relating log GLSM with the Gromov–Witten theory of complete intersections in $\mathcal{X}$.

As shown in [18], by imposing a certain stability condition (see §2.1.4), stable log R-maps with given discrete data $(g, \vec{\varsigma}, \beta)$ form a proper log Deligne-Mumford stack $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$. The discrete data consists of the genus $g \in \mathbb{N}$, the curve class $\beta \in H_2(\mathcal{X})$, and the discrete data at the markings $\vec{\varsigma} = \{(\overline{\gamma}_p, \mathbf{c}(p))\}_p$, which assigns to each marking p an orbifold sector $\overline{\gamma}_p$ parameterizing cyclotomic gerbes in $\mathfrak{P}_{\mathbf{k}} := \mathfrak{P} \times_{\mathbf{BC}^*_\omega} \operatorname{Spec} \mathbf{k}$ and a *contact order* $\mathbf{c}(p) \in \mathbb{N}$, such that f is required to be tangent to ∞ of order $\mathbf{c}(p)$. The most important case is when $(g, \vec{\varsigma}, \beta)$ is of *compact type*, which means that $\mathbf{c}(p) = 0$ and $\overline{\gamma}_p$ is a sector in the zero section $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}} = \mathbf{0}_{\mathfrak{P}} \times_{\mathbf{BC}^*_\omega} \operatorname{Spec} \mathbf{k}$ for all markings. In particular stable R-maps parameterized by $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$ factors through $\mathbf{0}_{\mathfrak{P}}$ along all markings.[1]

Similar to the case of Gromov–Witten theory, the smoothness of $\mathcal{X}$ leads to a *canonical perfect obstruction theory* of $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$, hence the *canonical virtual cycle* $[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{\mathrm{vir}}$, see (2.6). This canonical perfect obstruction theory solely depends on the geometry of $\mathfrak{P}$. To recover FJRW theory and the Gromov–Witten theory of complete intersections in $\mathcal{X}$, the extra key ingredient is a *superpotential*, that is a $\mathbb{C}^*_\omega$-equivariant holomorphic function $W \colon \mathfrak{P}^\circ_{\mathbf{k}} \to \mathbb{C}_\omega$, or equivalently a morphism of stacks $\mathfrak{P}^\circ \to \mathcal{L}_\omega$. A superpotential W is said to have *proper critical locus* if its critical locus is supported on $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$, see §4.1.

Given discrete data $(g, \vec{\varsigma}, \beta)$ of compact type, a superpotential W with proper critical loci can be used to modify the canonical theory of $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$, defining a reduced perfect obstruction theory hence the *reduced virtual cycle* $[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{\mathrm{red}}$ of $\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$, see [18] and §4. The reduced theory is of primary interest — depending on the choice of target $\mathfrak{P}$, it recovers both FJRW invariants, and Gromov–Witten invariants of complete intersections in $\mathcal{X}$.

1.2. **The tropical decomposition formula.** In this article, we study the structure of $[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{\mathrm{red}}$ from a tropical point of view. Let n be the number of markings labeled by $\vec{\varsigma}$. Consider the stack $\mathscr{M}_{g,n}(\mathcal{X}, \beta)$ of twisted stable maps. There is a tautological morphism

$$F_{\mathscr{M}} \colon \mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta) \to \mathscr{M}_{g,n}(\mathcal{X}, \beta)$$

obtained by composing stable log R-maps (1.1) with the canonical morphism $\mathfrak{P} \to \mathcal{X}$, removing the log structures and stabilizing.

Theorem 1.1 (“Tropical decomposition formula”, see Theorem 5.3)**.** *Let $[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{\mathrm{red}}$ be the reduced virtual cycle defined by the compact-type discrete data $(g, \vec{\varsigma}, \beta)$ and a superpotential W with proper critical locus. Then, we have the decomposition*

$$
\begin{aligned}
(1.2)\quad F_{\mathscr{M},*}[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)]^{\mathrm{red}} &= \sum_{\boldsymbol{\tau}_\curlywedge \vdash (g,\vec{\varsigma},\beta)} \frac{(-\tilde{r})^{|\mathbf{V}_\infty|}}{|\operatorname{Aut}(\boldsymbol{\tau}_\curlywedge)|} \cdot \prod_{E \in \mathbf{E}(G)} \mathbf{c}(E) \\
&\cdot F^G_{\mathscr{M},*} \circ \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{V \in \mathbf{V}_\infty(G)} [\mathscr{R}_{\mathbf{g}(V),\vec{\boldsymbol{\varsigma}}(V)}(\infty_{\mathfrak{P}}, \boldsymbol{\beta}(V))]^{\mathrm{red}} \times \prod_{V \in \mathbf{V}_0(G)} [\mathscr{R}_{\mathbf{g}(V),\vec{\boldsymbol{\varsigma}}(V)}(\mathfrak{P}, \boldsymbol{\beta}(V))]^{\mathrm{vir}} \right)
\end{aligned}
$$

where the sum ranges over partitions of $(g, \vec{\varsigma}, \beta)$ by decorated bipartite graphs

$$\boldsymbol{\tau}_\curlywedge = (G, \mathbf{V}(G) = \mathbf{V}_0(G) \sqcup \mathbf{V}_\infty(G), \mathbf{g}, \vec{\boldsymbol{\varsigma}}, \boldsymbol{\beta})\,. \tag{1.3}$$

We briefly explain the main result and refer to §5.1 and Theorem 5.3 for more details.

1.2.1. *Combinatorial partitions.* The partitions (1.3) naturally arise as decorated $\curlywedge$-types of a rigid tropical maps (5.1), hence the name “tropical decomposition”. As explained in §5.1.1, they are equivalent to the following combinatorial data.

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[1] The notation $\mathscr{R}^{\mathrm{cpt}}_{g,\vec{\varsigma}}(\mathfrak{P}, \beta)$ is used in [18] to emphasize the compact type condition of $(g, \vec{\varsigma}, \beta)$. In this paper we drop this supscript for simplicity noting that the compact type condition is built into $\vec{\varsigma}$.

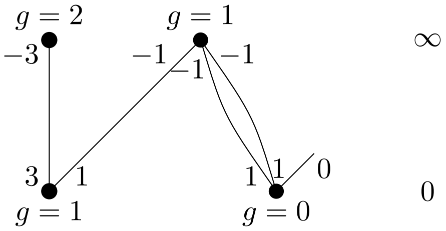

FIGURE 1. Example of a partition of $(g = 5, (0, 0_{\mathfrak{P}_\mathbf{k}}), 0)$ by a bipartite graph for the target $\mathfrak{P} = \mathbf{BC}^*_\omega$. We indicated the genus at each vertex, leaving out the curve classes, which are all zero. The numbers at the half-edges indicate the contact orders.

First, G is a non-empty connected graph consisting of the set of vertices $\mathbf{V}(G)$, the set of half edges $\mathbf{H}(G)$ and an involution $\iota_G\colon \mathbf{V}(G) \sqcup \mathbf{H}(G) \to \mathbf{V}(G) \sqcup \mathbf{H}(G)$ fixing $\mathbf{V}(G)$, see §B.1.6. As usual, we define

$$\mathbf{L}(G) = \{h \in \mathbf{H}(G) \mid \iota_G(h) = h\}, \qquad \mathbf{E}(G) = \{\{h, \hat{h}\} \mid h, \hat{h} \in \mathbf{H}(G) \mid \iota_G(h) = \hat{h} \neq h\},$$

to be the set of legs and the set of edges of G respectively. Furthermore, the legs $\mathbf{L}(G)$ correspond to the markings in $\vec{\varsigma}$ and are labeled $1, \ldots, n$.

The partition $\mathbf{V}(G) = \mathbf{V}_0(G) \sqcup \mathbf{V}_\infty(G)$ divides $\mathbf{V}(G)$ to a set $\mathbf{V}_0(G)$ of 0-*vertices* and a set $\mathbf{V}_\infty(G)$ of ∞-*vertices*. Note that both $\mathbf{V}_0(G)$ and $\mathbf{V}_\infty(G)$ are allowed to be empty, but not simultaneously.

The *genus decoration* $\mathbf{g}\colon \mathbf{V}(G) \to \mathbb{N}$ assigns a non-negative integer to each vertex, forming a partition of g in the following sense:

$$g = h^1(G) + \sum_{V \in \mathbf{V}(G)} \mathbf{g}(V).$$

The *curve class decoration* $\beta\colon \mathbf{V}(G) \to H_2(\mathcal{X})$ assigns a curve class to each vertex, forming a partition of β:

$$\beta = \sum_{V \in \mathbf{V}(G)} \boldsymbol{\beta}(V)$$

The data $\vec{\varsigma} = \{(\mathbf{c}(h), \overline{\gamma}_h)\}_{h \in \mathbf{H}(G)}$ specifies for each $h \in \mathbf{H}(G)$ a contact order $\mathbf{c}(h) \in \mathbb{Z}$ and a sector $\overline{\gamma}_h$ such that

(1) The set $\mathbf{L}(G)$ is labeled by the markings in $\vec{\varsigma}$ such that $\{(\mathbf{c}(h), \overline{\gamma}_h)\}_{h \in \mathbf{L}(G)} = \vec{\varsigma}$.
(2) For any edge $\{h, \hat{h}\} \in \mathbf{E}(G)$, the two sectors $\overline{\gamma}_h$ and $\overline{\gamma}_{\hat{h}}$ are ∞-sectors related by the nodal involution (2.11).
(3) For any edge $E = \{h, \hat{h}\} \in \mathbf{E}(G)$, we require $\mathbf{c}(h) = -\mathbf{c}(\hat{h})$, and define $\boldsymbol{c}(E) = |\mathbf{c}(h)| = |\mathbf{c}(\hat{h})|$.
(4) If $h \in \mathbf{H}(G)$ satisfying $\mathbf{c}(h) \geq 0$, then h is attached to a 0-vertex. Furthermore, we requre $\mathbf{c}(h) = 0$ iff $h \in \mathbf{L}(G)$.

In particular, half edges attached to ∞-vertices have strictly negative contact orders. We give an example of a partition by a bipartite graph in Figure 1.

1.2.2. *Vertex contributions.* For each vertex $V \in \mathbf{V}(G)$, denote by $\vec{\varsigma}(V) = \{(\mathbf{c}(h), \overline{\gamma}_h)\}_{h \in \mathbf{H}(V)}$ the subset of $\vec{\varsigma}$, where $\mathbf{H}(V) \subset \mathbf{H}(G)$ is the set of half edges attached to V. We distinguish the following two cases.

If $V \in \mathbf{V}_0(G)$, we obtain the stack $\mathscr{R}_{\mathbf{g}(V), \vec{\varsigma}(V)}(\mathfrak{P}, \boldsymbol{\beta}(V))$ of stable log R-maps with the corresponding discrete data, hence the canonical virtual cycle $[\mathscr{R}_{\mathbf{g}(V), \vec{\varsigma}(V)}(\mathfrak{P}, \boldsymbol{\beta}(V))]^{\mathrm{vir}}$.

If $V \in \mathbf{V}_\infty(G)$, then $\mathscr{R}_{\mathbf{g}(V), \vec{\varsigma}(V)}(\infty_{\mathfrak{P}}, \boldsymbol{\beta}(V))$ is the stack $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)$ of stable punctured R-maps associated to V as introduced in (5.5). Note that the half edges attached to V have negative contact orders in general, and hence a theory of punctured R-maps is necessary to describe these vertex moduli spaces. In this paper we will construct the stack of punctured R-maps in general. In particular, we will see that $\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}},\boldsymbol{\beta}(V))$ is a log DM stack admitting a reduced virtual cycle $[\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}},\boldsymbol{\beta}(V))]^{\mathrm{red}}$.

1.2.3. *Morphisms involved in the tropical decomposition.* For each edge $x=\{h,\hat{h}\}\in\mathbf{E}(G)$ with h attached to an ∞-vertex V and $\hat{h}$ attached to a 0-vertex $\hat{V}$, we obtain two evaluation morphisms

$$\mathrm{ev}_h\colon \mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}},\boldsymbol{\beta}(V)) \longrightarrow \overline{\gamma}_h,$$
$$\mathrm{ev}_{\hat{h}}\colon \mathscr{R}_{\mathbf{g}(\hat{V}),\vec{\varsigma}(\hat{V})}(\mathfrak{P},\boldsymbol{\beta}(\hat{V})) \longrightarrow \overline{\gamma}_{\hat{h}},$$

as in (2.16). These evaluation morphisms are determined by the orbifold structure of the underlying R-maps along the markings corresponding to h and $\hat{h}$. Set $\overline{\gamma}_x := \overline{\gamma}_h$, and consider the *involuted diagonal*

$$\Delta_x := Id\times\breve{\iota}\colon \overline{\gamma}_x \to \overline{\gamma}_h\times\overline{\gamma}_{\hat{h}},$$

see (5.8). Taking products, we obtain

$$\Delta_{\boldsymbol{\tau}_{\curlywedge}} := \prod_{x=\{h,\hat{h}\}\in\mathbf{E}(G)} \Delta_x\colon \prod_{x\in\mathbf{E}(G)}\overline{\gamma}_x \longrightarrow \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)}\overline{\gamma}_h\times\overline{\gamma}_{\hat{h}}.$$

Define the stack $\mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$ via the Cartesian diagram

$$\begin{tikzcd}
\mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge}) \arrow[r] \arrow[d] & \prod_{V\in\mathbf{V}_\infty(G)}\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}},\boldsymbol{\beta}(V))\times\prod_{V\in\mathbf{V}_0(G)}\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\mathfrak{P},\boldsymbol{\beta}(V)) \arrow[d] \\
\prod_{x\in\mathbf{E}(G)}\overline{\gamma}_x \arrow[r, "\Delta_{\boldsymbol{\tau}_{\curlywedge}}"'] & \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)}\overline{\gamma}_h\times\overline{\gamma}_{\hat{h}}
\end{tikzcd}$$

This stack $\mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$ carries a universal underlying R-map obtained by gluing universal R-maps from each vertex. Hence we obtain the tautological morphism

$$F^G_{\mathscr{M}}\colon \mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge}) \to \mathscr{M}_{g,n}(\mathcal{X},\beta),$$

defined similarly to $F_{\mathscr{M}}$.

This explains the morphisms involved in the tropical decomposition 1.2.

Remark 1.2. It should be emphasized that a similar formula replacing reduced virtual cycles by canonical virtual cycles in 1.2 is false. While this paper builds upon the theory of punctured maps [3], to establish 1.2 extra ingredients are required. One of the ingredient, discussed in §6, is a careful study of punctured R-maps to $\infty_{\mathfrak{P}}$ from *disconnected* domains. In particular, this gives rise to the special factor $(-\tilde{r})^{|\mathbf{V}_\infty|}$.

1.3. **Applications.** In Section 7.1, we specialize the tropical decomposition formula, Theorem 1.1, to the case of Gromov–Witten invariants of smooth hypersurfaces $\mathcal{Z}$ in smooth projective varieties $\mathcal{X}$. Let $\mathfrak{P}\to\mathbf{BC}^*_\omega$ be the log GLSM target corresponding to this geometry.

Corollary 1.3. *The Gromov–Witten invariants of $\mathcal{Z}$ with ambient insertions may be effectively computed in terms of two ingredients:*

1. *The* effective invariants *of $\mathcal{Z}\subset\mathcal{X}$, see* [19, §9].
2. *Tautological integrals against the* canonical *virtual cycle* $[\mathscr{R}_{g,n}(\mathfrak{P},\beta)]^{\mathrm{vir}}$.

The effective invariants are defined as integrals against the reduced virtual cycle of $\mathscr{R}_{g,n}(\infty_{\mathfrak{P}},\beta)$. The work [19] is dedicated to the axiomatic study of the effective invariants. In particular, it is shown that effective invariants vanish in many cases, and when they do not vanish, they often may be computed in terms of finitely many *basic effective invariants* in each genus. On the other hand, we expect the non-vanishing effective invariants to be the most challenging part of computing higher genus Gromov–Witten invariants of hypersurfaces, and more generally, complete intersections.

By [18, Corollary 1.10], the Gromov–Witten invariants of $\mathcal{Z}$ are equal to integrals against the *reduced* virtual cycle of $\mathscr{R}_{g,n}(\mathfrak{P}, \beta)$. On the other hand, the second ingredient in Corollary 1.3 is formed by the integrals against the *canonical* virtual cycle of the same moduli space.

The canonical virtual cycle is not only much easier to define than the reduced one but also has significant advantages from the computational point of view. One of the advantages of the canonical perfect obstruction theory is that, as discussed in [17], it often is equivariant for a larger torus than the reduced perfect obstruction theory. For instance, if $\mathcal{Z}$ is a quintic threefold in $\mathbb{P}^4$, then the canonical perfect obstruction theory is equivariant with respect to a $(\mathbb{C}^*)^5$ acting on the base $\mathbb{P}^4$ as well as a $\mathbb{C}^*_\omega$ acting by scaling R-maps in the projective bundle direction. In contrast, any suitable superpotential, and thus the reduced perfect obstruction theory, is only equivariant for the latter $\mathbb{C}^*_\omega$-action (also called the *R-action*).

The additional equivariance is used crucially in the proof [25] of several predictions from physics about the structure of higher genus Gromov–Witten invariants of quintic threefolds, including Yamaguchi–Yau's finite generation (see also §1.4 below) and the holomorphic anomaly equations. More specifically, combining the tropical decomposition formula with the localization formula of [17] for the canonical virtual cycle, allows to compute the Gromov–Witten invariants of $\mathcal{Z}$ in terms of the $(\mathbb{C}^*)^5 \times \mathbb{C}^*_\omega$-equivariant $\mathcal{O}(5)$-twisted Gromov–Witten invariants of $\mathbb{P}^4$. Having the $(\mathbb{C}^*)^5 \times \mathbb{C}^*_\omega$-equivariance, then allows to make the following specialization of equivariant parameters

$$(\lambda_0, \lambda_1, \dots, \lambda_4, t) = (\lambda, \zeta\lambda, \dots, \zeta^4\lambda, 0),$$

where λ is a single equivariant parameter, and ζ is a primitive 5th root of unity. Under this specialization, the twisted Gromov–Witten theory recovers the "formal quintic" in the sense of Lho–Pandharipande [28], for which finite generation and holomorphic anomaly are well-understood.

A second advantage of the canonical perfect obstruction theory is that, as its name suggests, it is the canonical obstruction theory from the point of view of logarithmic Gromov–Witten theory, and we should thus expect many results from log Gromov–Witten theory to translate to the log R-map setting. One specific result we have in mind is Tseng–You's work [35], which computes the log Gromov–Witten invariants of a smooth projective variety X with divisorial log structure along a smooth divisor D in terms of orbifold Gromov–Witten invariants of the root stacks $\sqrt[r]{(X, D)}$. Applying a similar approach to log R-maps has the potential to provide a uniform treatment of Yamaguchi–Yau's finite generation and the holomorphic anomaly equations even in cases where a "formal theory" does not exist. We hope to explore this direction in future works.

We conclude by pointing out log GLSM is only one of several approaches to the structure of higher genus Gromov–Witten invariants of quintic threefolds. One other approach is via the technique of mixed spin p-fields of Chang–Guo–Li–Li–Liu [10, 11, 12, 14, 15].

1.4. **Connection to mirror symmetry.** We outline how even by itseflf, the tropical decomposition formula interacts nicely with higher genus mirror symmetry, in particular with Yamaguchi–Yau's finite generation property.

We begin by recalling two generating series of the Gromov–Witten invariants of a Calabi–Yau threefold $\mathcal{Z}$. The genus $g \geq 2$ Gromov–Witten invariants of $\mathcal{Z}$ may be assembled into a generating series

$$F_g(Q) = \sum_\beta Q^\beta \deg[\mathscr{M}_{g,n}(\mathcal{Z}, \beta)]^{\mathrm{vir}} \in \Lambda,$$

which is valued in the Novikov ring Λ. The series $F_g(Q)$ is called the *genus g Gromov–Witten potential of $\mathcal{Z}$*.

For genus $g = 0$, we instead consider the *(small) J-function of $\mathcal{Z}$*

$$J(Q,z) = z \cdot \mathbf{1} + \sum_{\beta \neq 0} Q^\beta \operatorname{ev}_{1*}\left(\frac{1}{z - \psi_1} \cap [\mathscr{M}_{0,1}(\mathcal{Z},\beta)]^{\mathrm{vir}} \right) \in H^*(\mathcal{Z}) \otimes \Lambda[z^{-1}],$$

which contains the information of all genus zero Gromov–Witten invariants of $\mathcal{Z}$.

Conjecture 1.4 (Genus zero mirror theorem)**.** *There exists a "mirror Calabi–Yau threefold" of $\mathcal{Z}$ whose associated series of periods I has an expansion*

$$I(q,z) = zI_0(q) + I_1(q) + O(z^{-1}) \in zH^*(\mathcal{Z}) \otimes \Lambda'[z^{-1}]$$

where Λ' is a Novikov ring in a second set of variables q^β, where $I_0(q) \in \Lambda'$ and $I_1(q) \in H^2(\mathcal{Z}) \otimes \Lambda'$.

Furthermore, the J-function may be computed from the I-function via the formula

$$J(Q,z) = \exp\left(-\frac{I_1}{zI_0} \right) \frac{I(q,z)}{I_0(q)}$$

under the "mirror map"

$$Q^\beta = q^\beta \exp\left(\int_\beta \frac{I_1(q)}{I_0(q)} \right) \tag{1.4}$$

isomorphism $\Lambda \cong \Lambda'$.

There is an extensive literature on special cases cases of the genus zero mirror conjecture. Here, we contends ourselves with mentioning Lian–Liu–Yau's proof [29] in the case of complete intersections in projective space and Givental's proof [22] in the case of complete intersections in toric varieties. Furthermore, Coates–Givental have developed a "quantum Lefschetz principle", which implies a general genus zero mirror theorem for hypersurfaces, see [21, Corollary 7].

Higher genus mirror symmetry concerns formulas for $F_g(Q)$ for $g \geq 2$ similar to Conjecture 1.4. The following is the fundamental structural prediction for $F_g(Q)$.

Conjecture 1.5 (Yamaguchi–Yau polynomiality/finite generation [36])**.** *Assume that Conjecture 1.4 holds for $\mathcal{Z}$. Then, there exists a finitely generated $\mathbb{Q}$-algebra $\mathcal{R} \subset \Lambda'$ such that*

$$F_g(Q) \in (I_0(q))^{2g-2}\mathcal{R}$$

for every $g \geq 2$.

We may apply the tropical decomposition formula for log GLSM to Yamaguchi–Yau's finite generation. We now let $\mathcal{Z}$ be a smooth Calabi–Yau hypersurface in a Fano fourfold $\mathcal{X}$. Let $\mathfrak{P} \to \mathbf{BC}^*_\omega$ be the corresponding log GLSM target. Form the following generating series

$$K(Q) = \sum_{\beta \neq 0} Q^\beta \operatorname{ev}_{1,*}\left([\mathscr{R}_{0,\vec{\varsigma}}(\mathfrak{P},\beta)]^{\mathrm{vir}} \right) \in H^2(\mathcal{X}) \otimes \Lambda,$$

where $\vec{\varsigma}$ consists of a single ∞-sector with contact order 1, and for any $g, n, l \geq 0$ such that $2g - 2 + 2n + 3l > 0$ and any $\alpha_1, \ldots, \alpha_n \in H^6(\mathcal{X})$, we define

$$\Omega_{g,n,l}(\alpha_1, \ldots, \alpha_n) = \sum_\beta Q^\beta \int_{[\mathscr{R}_{g,\vec{\varsigma}_{n+l}}(\mathfrak{P},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \operatorname{ev}_i^*(\alpha_i) \prod_{i=n+1}^{n+l} \operatorname{ev}_i^*(p) \in \Lambda,$$

where $\vec{\varsigma}_{n+l}$ consists of n many ∞-sectors with contact order 1 and l many ∞-sectors with contact order 2, and $p \in H^8(\mathcal{X})$ denotes the Poincaré dual class of a point. We expect that these series satisfy the following Yamaguchi–Yau-type polynomiality:

Conjecture 1.6. *Assume that Conjecture 1.4 holds for $\mathcal{Z}$. Then, under the mirror map (1.4), we have*

$$K(Q) = \log(I_0(q)) - \frac{I_1(q)}{I_0(q)},$$

and for all g, n, l such that $2g - 2 + 2n + 3l > 0$ and $\alpha_1, \dots, \alpha_n$, we have

$$\Omega_{g,n,l}(\alpha_1, \dots, \alpha_n) \in (I_0(q))^{2g-2+2n+3l}\mathcal{R}$$

for a finitely generated $\mathbb{Q}$-algebra $\mathcal{R} \subseteq \Lambda'$.

Theorem 1.7. *If Conjecture 1.6 holds for $\mathcal{Z}$ and we have $q^\beta \in \mathcal{R}$ for every effective curve class β, then Conjecture 1.5 holds for $\mathcal{Z}$.*

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2. PUNCTURED R-MAPS AND THE CANONICAL PERFECT OBSTRUCTION THEORY

2.1. **Punctured R-maps.**

Log R-maps are at the heart of log GLSM [18]. To compute their boundary contributions in log GLSM, we combine the theory of punctured logarithmic maps [3] and R-structure to extend the notions of log R-maps to punctured R-maps, which allow negative contact orders along markings. For reader's convenience, a review of [3] in the setting needed in this paper is provided in the appendix §B.

2.1.1. *Hybrid targets.* The *hybrid targets* of [18] are targets for both log R-maps and punctured R-maps of this paper. They are constructed using the following data:

(1) A proper Deligne–Mumford stack $\mathcal{X}$ with a projective coarse moduli scheme X.
(2) A vector bundle $\mathbf{E}$ over $\mathcal{X}$ of the form

$$\mathbf{E} = \bigoplus_{i \in \mathbb{Z}_{>0}} \mathbf{E}_i$$

where $\mathbf{E}_i$ is a vector bundle with the grading i. Write $d := \gcd\big(i \mid \mathbf{E}_i \neq 0\big)$.
(3) A line bundle $\mathbf{L}$ over $\mathcal{X}$.
(4) A positive integer r.

For later use, fix an ample line bundle H over X, and denote by $\mathcal{H}$ its pull-back to $\mathcal{X}$.

Denote by $\mathbf{BC}^*_\omega := \mathbf{BG}_m$ with the trivial log structure and the universal line bundle $\mathcal{L}_\omega$. The inertial torus of $\mathbf{BC}^*_\omega$ will be denoted by $\mathbb{C}^*_\omega$. The data $(\mathcal{X}, \mathbf{L}, r)$ leads to a cartesian diagram

(2.1)
$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\mathcal{L}_{\mathfrak{X}}"] \arrow[d] & \mathbf{BG}_m \arrow[d, "\nu_r"] \\
\mathbf{BC}^*_\omega \times \mathcal{X} \arrow[r, "\mathcal{L}_\omega \boxtimes \mathbf{L}^\vee"] & \mathbf{BG}_m,
\end{tikzcd}$$

defining the stack $\mathfrak{X}$, where ν_r is the rth power map, the bottom arrow is defined by $\mathcal{L}_\omega \boxtimes \mathbf{L}^\vee$, and the top arrow is defined by $\mathcal{L}_{\mathfrak{X}}$ — the universal r-th root of $\mathcal{L}_\omega \boxtimes \mathbf{L}^\vee$, called the *$r$-spin structure.*

To construct the target $\mathfrak{P}$ (the symbol $\mathfrak{P}$ is the fraktur letter "P"), we fix a *twisting choice* $a \in \frac{1}{d} \cdot \mathbb{Z}_{>0}$, or equivalently a rational number $\tilde{r} = a \cdot r$. The underlying $\underline{\mathfrak{P}}$ is the weighted projective stack bundle over $\mathfrak{X}$:

$$\underline{\mathfrak{P}} := \underline{\mathbb{P}}^{\mathbf{w}}\left(\bigoplus_{i>0}(\mathbf{E}_{i,\mathfrak{X}}^{\vee}\otimes\mathcal{L}_{\mathfrak{X}}^{\otimes i})\oplus\mathcal{O}_{\mathfrak{X}}\right), \tag{2.2}$$

with $\mathbf{w}$ the collection of the weights of the $\mathbb{G}_m$-action such that the weight on the i-th factor is the positive integer $a\cdot i$, while the weight of the last factor $\mathcal{O}_{\mathfrak{X}}$ is 1. Here, for any vector bundle $V=\oplus_i V_i$ with a $\mathbb{G}_m$-action of weight $\mathbf{w}$, we use the notation

$$\mathbb{P}^{\mathbf{w}}(V) = \left[\Big(\mathrm{Vb}(V)\setminus\mathbf{0}_V\Big)\Big/\mathbb{G}_m\right], \tag{2.3}$$

where $\mathrm{Vb}(V)$ is the total space of V, and $\mathbf{0}_V$ is the zero section of $\mathrm{Vb}(V)$.

Note $\underline{\mathfrak{P}}$ is a compactification of

$$\mathfrak{P}^{\circ} := \mathrm{Vb}\left(\bigoplus_{i>0}(\mathbf{E}_{i,\mathfrak{X}}^{\vee}\otimes\mathcal{L}_{\mathfrak{X}}^{\otimes i})\right)\subset\underline{\mathfrak{P}}. \tag{2.4}$$

The boundary $\infty_{\mathfrak{P}}=\underline{\mathfrak{P}}\setminus\mathfrak{P}^{\circ}$ is the Cartier divisor defined by the vanishing of the $\mathcal{O}_{\mathfrak{X}}$-coordinate in (2.2). We make $\underline{\mathfrak{P}}$ into a log stack $\mathfrak{P}$ by equipping it with the log structure corresponding to the Cartier divisor $\infty_{\mathfrak{P}}$. In particular, there is a canonical strict morphism of log stacks

$$\mathfrak{P}\to\mathcal{A}, \tag{2.5}$$

see §A.2.1. Furthermore, we have the following commutative diagram

$$\begin{tikzcd} \mathfrak{P} \arrow[r, "\mathfrak{p}"] \arrow[dr, "\mathfrak{t}"'] & \mathfrak{X} \arrow[r, "\zeta"] \arrow[d] & \mathbf{BC}^{*}_{\omega} \\ & \mathcal{X} & \end{tikzcd} \tag{2.6}$$

where ζ is the composition $\mathfrak{X}\to\mathbf{BC}^{*}_{\omega}\times\mathcal{X}\to\mathbf{BC}^{*}_{\omega}$ with the second arrow the projection to $\mathbf{BC}^{*}_{\omega}$. By construction, $\zeta\circ\mathfrak{p}$ is proper of DM-type. For later use, denote by $\mathbf{0}_{\mathfrak{P}}$ the zero section of the vector bundle $\mathfrak{P}^{\circ}$.

2.1.2. *Punctured R-maps.* Fix a target $\mathfrak{P}\to\mathbf{BC}^{*}_{\omega}$ as in (2.6). A *punctured R-map* over an fs log scheme S is a 2-commutative triangle

$$\begin{tikzcd} & \mathfrak{P} \arrow[d, "\zeta\circ\mathfrak{p}"] \\ C^{\circ} \arrow[ur, "f"] \arrow[r, "\omega^{\log}_{C^{\circ}/S}"'] & \mathbf{BC}^{*}_{\omega} \end{tikzcd} \tag{2.7}$$

where $C^{\circ}\to S$ is a punctured curve as in (B.4), f is a morphism of log stacks, and the bottom morphism and the 2-morphism define an isomorphism $\omega^{\log}_{C^{\circ}/S}\cong\mathcal{L}_{\omega}|_{C^{\circ}}$. It is called a *log R-map* if $C^{\circ}\to S$ is a family of log curves.

For simplicity, we may denote a punctured R-map by $f\colon C^{\circ}\to\mathfrak{P}$. Removing log structures from (2.7), we obtain the corresponding *underlying R-map* $\underline{f}\colon\underline{C}\to\underline{\mathfrak{P}}$ over $\underline{S}$.

2.1.3. *The associated punctured maps to $\mathcal{A}$.* For a punctured R-map $f\colon C^{\circ}\to\mathfrak{P}$ over S, further composing with (2.5) we obtain the *associated punctured map* $\mathfrak{f}\colon C^{\circ}\longrightarrow\mathcal{A}$ over S. Since the arrow $\mathfrak{P}\to\mathcal{A}$ is strict, the logarithmic data of f is encoded in $\mathfrak{f}$. We call f *pre-stable* if $\mathfrak{f}$ is pre-stable as in §B.2.2. Unless otherwise specified, punctured R-maps we consider are always assumed to be pre-stable.

The *tropical type* of f is defined to be the tropical type of $\mathfrak{f}$ as in (B.19). A punctured R-map is said to be marked by a type τ if its associated punctured map to $\mathcal{A}$ is marked by τ, see §B.3.3. We call f *basic* if $\mathfrak{f}$ is basic.

2.1.4. *Stability.* A punctured R-map $f\colon C^\circ \to \mathfrak{P}$ over S is *stable* if it is representable and satisfies the following positivity condition

$$(\omega^{\log}_{C^\circ/S})^{1+\delta} \otimes (\mathfrak{t}\circ f)^*\mathcal{H}^{\otimes k} \otimes f^*\mathcal{O}(\tilde{r}\infty_{\mathfrak{P}}) > 0, \tag{2.8}$$

for $k \gg 0$ and arbitrarily small $\delta > 0$. This is identical to [18, §1.3.4] for log R-maps.

2.2. **Orbifold sectors along markings and nodes.** Next we discuss the constraints from orbifold structures along nodes and markings of punctured R-maps. They impose important conditions on gluings of underlying R-maps.

2.2.1. *An involution of the target.* Consider the base change

$$\mathfrak{P}_{\mathbf{k}} := \mathfrak{P} \times_{\mathbf{BC}^*_\omega} \operatorname{Spec}\mathbf{k}, \qquad \mathfrak{X}_{\mathbf{k}} := \mathfrak{X} \times_{\mathbf{BC}^*_\omega} \operatorname{Spec}\mathbf{k}$$

along the quotient $\operatorname{Spec}\mathbf{k} \to \mathbf{BC}^*_\omega$ by $\mathbb{C}^*_\omega$. This leads to a natural $\mathbb{C}^*_\omega$-action on both $\mathfrak{P}_{\mathbf{k}}$ and $\mathfrak{X}_{\mathbf{k}}$ making the projection $\mathfrak{P}_{\mathbf{k}} \to \mathfrak{X}_{\mathbf{k}}$ equivariant. This $\mathbb{C}^*_\omega$-action is called the *R-action*, and will be studied in more detail in [17].

Consider the strict closed substacks of $\mathfrak{P}_{\mathbf{k}}$

$$\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}} := \mathbf{0}_{\mathfrak{P}} \times_{\mathbf{BC}^*_\omega} \operatorname{Spec}\mathbf{k}, \qquad \infty_{\mathfrak{P}_{\mathbf{k}}} := \infty_{\mathfrak{P}} \times_{\mathbf{BC}^*_\omega} \operatorname{Spec}\mathbf{k}.$$

Note that the R-action on $\mathfrak{P}_{\mathbf{k}}$ naturally restricts to a $\mathbb{C}^*_\omega$-action on $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ and $\infty_{\mathfrak{P}_{\mathbf{k}}}$, since they are stable under the R-action. Multiplication by $-1 \in \mathbb{C}^*_\omega$ via the R-action defines a strict involution

$$\iota_\omega\colon \mathfrak{P}_{\mathbf{k}} \to \mathfrak{P}_{\mathbf{k}}, \tag{2.9}$$

restricting to involutions $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}} \to \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ and $\infty_{\mathfrak{P}_{\mathbf{k}}} \to \infty_{\mathfrak{P}_{\mathbf{k}}}$, denoted again by ι_ω.

Remark 2.1. As discussed above and in [17, §3], both $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ and $\infty_{\mathfrak{P}_{\mathbf{k}}}$ are fixed loci of the $\mathbb{C}^*_\omega$-action. Despite this, in case of non-trivial stacky structure of $\mathfrak{P}_{\mathbf{k}}$, the $\mathbb{C}^*_\omega$-action on $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ and $\infty_{\mathfrak{P}_{\mathbf{k}}}$ is only trivial after passing to a reparameterization of $\mathbb{C}^*_\omega$. As a consequence, the involutions $\iota_\omega\colon \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}} \to \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ and $\iota_\omega\colon \infty_{\mathfrak{P}_{\mathbf{k}}} \to \infty_{\mathfrak{P}_{\mathbf{k}}}$ are 2-isomorphic to the identity, however not canonically unless $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ and $\infty_{\mathfrak{P}_{\mathbf{k}}}$ are schemes, see [17, Corollary 3.8(c)].

It follows that one may identify ι_ω with the identity, but doing so would lead to diagrams that are 2-commutative in a non-canonical way. On the contrast, in this paper, we will not identify ι_ω with the identity, and in this way, all 2-commutative diagrams that we consider have a canonical 2-morphism.

2.2.2. *Orbifold sectors.* Since $\zeta\circ\mathfrak{p}$ is proper of DM-type, the underlying $\underline{\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}}$, $\underline{\infty_{\mathfrak{P}_{\mathbf{k}}}}$ and $\underline{\mathfrak{P}_{\mathbf{k}}}$ are proper DM-stacks. Denote by $\overline{\mathcal{I}}_\mu\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}, \overline{\mathcal{I}}_\mu\infty_{\mathfrak{P}_{\mathbf{k}}}$ and $\overline{\mathcal{I}}_\mu\mathfrak{P}_{\mathbf{k}}$ the rigidified inertia stacks of the corresponding underlying stacks as in [5], with their corresponding universal cyclotomic gerbes

$$\mathcal{I}_\mu\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}} \to \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}, \qquad \mathcal{I}_\mu\infty_{\mathfrak{P}_{\mathbf{k}}} \to \infty_{\mathfrak{P}_{\mathbf{k}}}, \qquad \mathcal{I}_\mu\mathfrak{P}_{\mathbf{k}} \to \mathfrak{P}_{\mathbf{k}}.$$

Here we equip $\mathcal{I}_\mu\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}, \mathcal{I}_\mu\infty_{\mathfrak{P}_{\mathbf{k}}}$ and $\mathcal{I}_\mu\mathfrak{P}_{\mathbf{k}}$ with the log structures pulled back from $\mathfrak{P}_{\mathbf{k}}$, and equip $\overline{\mathcal{I}}_\mu\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}, \overline{\mathcal{I}}_\mu\infty_{\mathfrak{P}_{\mathbf{k}}}$ and $\overline{\mathcal{I}}_\mu\mathfrak{P}_{\mathbf{k}}$ with trivial log structures.

Definition 2.2. A 0-*sector* (resp. ∞-*sector* or $\mathfrak{P}$-*sector*) is an irreducible component $\overline{\gamma} \subset \overline{\mathcal{I}}_\mu\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ (resp. $\overline{\gamma} \subset \overline{\mathcal{I}}_\mu\infty_{\mathfrak{P}_{\mathbf{k}}}$ or $\overline{\gamma} \subset \overline{\mathcal{I}}_\mu\mathfrak{P}_{\mathbf{k}}$).[2]

In the following, by a *sector* $\overline{\gamma}$ we mean either a 0-sector, an ∞-sector or a $\mathfrak{P}$-sector. For a sector $\overline{\gamma}$ with the universal gerbe $ev_{\overline{\gamma}}\colon \gamma \to \mathfrak{P}_{\mathbf{k}}$, denote by $\frac{1}{|\gamma|}$ the degree of $\gamma \to \overline{\gamma}$. Thus $\gamma \to \overline{\gamma}$ is a $\mu_{|\gamma|}$-gerbe. Note that γ is equipped with the log structure pulled back from $\mathfrak{P}_{\mathbf{k}}$.

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[2] In [19, Definition 2.13], 0-sectors are also called *sectors of compact type*, and a sector is called a *log sector* if it is either an ∞-sector or a $\mathfrak{P}$-sector.

The involution (2.9) induces a natural involution on the set of sectors as follows. Let $\overline{\gamma}$ be a sector with the universal gerbe $\boldsymbol{ev}_{\overline{\gamma}}\colon \gamma \to \mathfrak{P}_{\mathbf{k}}$. We consider the morphisms

$$\gamma' \xleftarrow{\iota'} \gamma \xrightarrow{\boldsymbol{ev}_\gamma} \mathfrak{P}_{\mathbf{k}} \xrightarrow{\iota_\omega} \mathfrak{P}_{\mathbf{k}}$$

where ι' is an isomorphism of stacks inverting the band over $\overline{\gamma}$. This leads to another family of gerbes $\boldsymbol{ev}_{\gamma'} := \iota_\omega \circ \boldsymbol{ev}_\gamma \circ (\iota')^{-1}\colon \gamma' \to \mathfrak{P}_{\mathbf{k}}$ over $\overline{\gamma}$. If $\overline{\gamma}$ is a 0-sector (resp. ∞-sector or $\mathfrak{P}$-sector), then by [5, §3] there is a unique 0-sector (resp. ∞-sector or $\mathfrak{P}$-sector) $\overline{\gamma}^{-1}$ with the universal gerbe $\boldsymbol{ev}_{\gamma^{-1}}\colon \gamma^{-1} \to \mathfrak{P}_{\mathbf{k}}$ fitting in a commutative diagram

(2.10)
$$\begin{tikzcd}
\gamma' \arrow[r] \arrow[d] \arrow[rr, bend left, "\boldsymbol{ev}_{\gamma'}"] & \gamma^{-1} \arrow[r, "\boldsymbol{ev}_{\gamma^{-1}}"'] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \\
\overline{\gamma} \arrow[r, "\breve{\iota}"] & \overline{\gamma}^{-1} &
\end{tikzcd}$$

where the square on the left is Cartesian.

Proposition 2.3. *The arrow $\breve{\iota}$ is an isomorphism fitting in a commutative diagram*

(2.11)
$$\begin{tikzcd}
\mathfrak{P}_{\mathbf{k}} \arrow[d, "\iota_\omega"'] & \gamma \arrow[l, "\boldsymbol{ev}_\gamma"'] \arrow[r] \arrow[d, "\iota_\gamma"] & \overline{\gamma} \arrow[d, "\breve{\iota}"] \\
\mathfrak{P}_{\mathbf{k}} & \gamma^{-1} \arrow[l, "\boldsymbol{ev}_{\gamma^{-1}}"'] \arrow[r] & \overline{\gamma}^{-1}
\end{tikzcd}$$

where the square on the right is Cartesian. Furthermore, both arrows ι_γ and $\breve{\gamma}$ are strict involutions. We will call $\breve{\iota}$ the nodal involution.

Proof. Observe a commutative diagram

$$\begin{tikzcd}
\gamma' \arrow[r] \arrow[dr, "(\iota')^{-1}"'] & \gamma^{-1} \arrow[r, "\boldsymbol{ev}_{\gamma^{-1}}"] & \mathfrak{P}_{\mathbf{k}} \\
 & \gamma \arrow[r, "\boldsymbol{ev}_\gamma"] & \mathfrak{P}_{\mathbf{k}} \arrow[u, "\iota_\omega"']
\end{tikzcd}$$

Thus, if we apply the same discussion with $(\overline{\gamma}, \boldsymbol{ev}_\gamma)$ replaced by $(\overline{\gamma}^{-1}, \boldsymbol{ev}_{\gamma^{-1}})$, we obtain the inverse $\overline{\gamma}^{-1} \to \overline{\gamma}$ of $\breve{\iota}$. □

2.2.3. *Sectors along half-edges.* Consider a representable punctured R-map $f\colon C^\circ \to \mathfrak{P}$ over a connected fs base S with the domain underlying curve marked by $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ as in §B.1.8. For a half-edge $h \in \mathbf{H}(G)$, let $p_h \to C^\circ$ be the strict closed embedding given by Notation B.2. We have two possibilities.

If $h \in \mathbf{L}(G)$ is a leg, then $p_h \subset C^\circ$ is the corresponding marking. The canonical residue isomorphism $\omega^{\log}_{C/S}|_{p_h} \cong \mathcal{O}_{p_h}$ induces a natural commutative diagram

(2.12)
$$\begin{tikzcd}
p_h \arrow[r, "\boldsymbol{ev}_h"] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\
C^\circ \arrow[r, "f"] & \mathfrak{P}
\end{tikzcd}$$

Furthermore, the representability of f implies that $\boldsymbol{ev}_h$ is also representable.

If $x = \{h, \hat{h}\} \in \mathbf{E}(G)$ forms an edge, let $\widetilde{C^\circ} \to C^\circ$ be the strict partial normalization along the node $p_x \subset C^\circ$. Consider the composition $\tilde{f}\colon \widetilde{C^\circ} \to C^\circ \xrightarrow{f} \mathfrak{P}$. It is representable since f is so. Similar to the case of legs, the canonical isomorphisms

$$\omega^{\log}_{C/S}|_{p_h} \xrightarrow{\cong} \omega^{\log}_{\widetilde{C}/S}|_{p_h} \xrightarrow{\cong} \mathcal{O}_{p_h}, \qquad \omega^{\log}_{C/S}|_{p_{\hat{h}}} \xrightarrow{\cong} \omega^{\log}_{\widetilde{C}/S}|_{p_{\hat{h}}} \xrightarrow{\cong} \mathcal{O}_{p_{\hat{h}}} \tag{2.13}$$

induce a pair of canonical commutative diagrams

$$\begin{tikzcd} p_h \arrow[r, "\mathit{ev}_h"] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\ \widetilde{C}^\circ \arrow[r, "\widetilde{f}"] & \mathfrak{P} \end{tikzcd} \qquad \begin{tikzcd} p_{\hat{h}} \arrow[r, "\mathit{ev}_{\hat{h}}"] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\ \widetilde{C}^\circ \arrow[r, "\widetilde{f}"] & \mathfrak{P} \end{tikzcd} \tag{2.14}$$

where both ev_h and $\mathit{ev}_{\hat{h}}$ are representable by the representability of $\widetilde{f}$. Note that the fibers (2.13) along p_h and $p_{\hat{h}}$ are canonically isomorphic $\omega^{\log}_{C/S}|_{p_h} \cong \omega^{\log}_{C/S}|_{p_{\hat{h}}}$ by multiplication by -1. The two morphisms ev_h and $\mathit{ev}_{\hat{h}}$ are related by the following commutative diagram

$$\begin{tikzcd} \mathfrak{P}_{\mathbf{k}} \arrow[d, "\iota_\omega"'] & p_h \arrow[l, "\mathit{ev}_h"'] \arrow[r] \arrow[d, "\iota_h"] & S \arrow[d, equal] \\ \mathfrak{P}_{\mathbf{k}} & p_{\hat{h}} \arrow[l, "\mathit{ev}_{\hat{h}}"] \arrow[r] & S, \end{tikzcd} \tag{2.15}$$

where ι_h is the isomorphism inverting the band (B.9).

For each $h \in \mathbf{H}(G)$, we are interested in one of the following cases:

(1) ev_h factors through $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$.
(2) ev_h factors through $\infty_{\mathfrak{P}_{\mathbf{k}}}$.
(3) ev_h factors through $\mathfrak{P}_{\mathbf{k}}$.

By [5, §3], in all three cases the connectedness of S implies a commutative diagram

$$\begin{tikzcd} p_h \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathit{ev}_h"] & \gamma_h \arrow[r] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \\ S \arrow[r, "\mathrm{ev}_h"] & \overline{\gamma}_h & \end{tikzcd} \tag{2.16}$$

for $\overline{\gamma}_h$ a 0-sector, ∞-sector or $\mathfrak{P}$-sector respectively. We call $\overline{\gamma}_h$ the *sector associated to* h.

Now assume $x = \{h, \hat{h}\} \in \mathbf{E}(G)$. As both ev_h and $\mathit{ev}_{\hat{h}}$ are induced by the restriction $f|_{p_x}$, we will assume that if $\overline{\gamma}_h$ is a 0-sector, ∞-sector or $\mathfrak{P}$-sector, then $\overline{\gamma}_{\hat{h}}$ should be a sector of the same type. By Proposition 2.3 and (2.15) we observe that the two sectors $\overline{\gamma}_h, \overline{\gamma}_{\hat{h}}$ are naturally related by the nodal involution

$$\breve{\iota}\colon \overline{\gamma}_h \to \overline{\gamma}_{\hat{h}} \tag{2.17}$$

Remark 2.4. In Case (1), we call that f is of *compact type* along h. In this case we necessarily have $\mathbf{c}(h) = 0$. Compact type condition along legs will be an important ingredient in the construction of the reduced theory in §4.

In Case (2), we will be mainly interested in the case that $\mathbf{c}(h) \neq 0$, which automatically forces the factorization of ev_h through $\infty_{\mathfrak{P}_{\mathbf{k}}}$.

In Case (3), it is natural to assume that $\mathbf{c}(h) = 0$, as otherwise ev_h factors through $\infty_{\mathfrak{P}_{\mathbf{k}}}$ as above. Case (3) is neither directly involved in the tropical decomposition formula, Theorem 1.1, nor the virtual localization for log GLSM [17]. However, we expect that a better understanding of this case will be crucial for a cohomological field theory structure for log GLSM. In this paper we will construct both the canonical and reduced perfect obstruction theory with nodal Case (3) sectors, and will leave further study of this case for further work.

2.3. **Combinatorial data of punctured R-maps.**

2.3.1. *Decorated types.* A *decorated type* consists of

$$\boldsymbol{\tau} = (\tau, \overline{\gamma}, \boldsymbol{\beta}) \tag{2.18}$$

where $\tau = (\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}), \boldsymbol{\sigma}, \mathbf{c})$ is a type as in (B.19) together with additional data:

1. The *sector decoration* $\overline{\gamma}$ assigns to each half-edge $h \in \mathbf{H}(G)$ a sector $\overline{\gamma}_h$ as in §2.2.2.
2. The *curve class decoration* $\boldsymbol{\beta}\colon \mathbf{V}(G) \to H_2(\mathcal{X})$ assigns to each $V \in \mathbf{V}(G)$ an effective curve class $\boldsymbol{\beta}(V)$.

We further require τ and $\overline{\gamma}$ are compatible in the following sense:

- (i) If $x = \{h, \hat{h}\} \in \mathbf{E}(G)$, then the pair $\overline{\gamma}_x := (\overline{\gamma}_h, \overline{\gamma}_{\hat{h}})$ are related by the nodal involution (2.17).
- (ii) $\mathbf{deg}(h) = \frac{1}{|\gamma_h|}$ for each $h \in \mathbf{H}(G)$.
- (iii) For any $h \in \mathbf{H}(G)$ with $\mathbf{c}(h) \neq 0$, $\overline{\gamma}_h$ is an ∞-sector.
- (iv) For a non-degenerate half-edge $h \in \mathbf{H}(G)$, $\overline{\gamma}_h$ is either a 0-sector or a $\mathfrak{P}$-sector.

A decorated type (2.18) is of *compact type* if for any $h \in \mathbf{L}(G)$ one of the following hold: either $\mathbf{c}(h) = 0$ and $\overline{\gamma}_h$ is a 0-sector; or $\mathbf{c}(h) < 0$. This compact type condition is necessary in the construction of the reduced theory in §4.

2.3.2. *Decorated tropical types.* Consider a stable punctured R-map $f\colon C^\circ \to \mathfrak{P}$ over an fs base S with $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ the associated punctured map to $\mathcal{A}$.

For each geometric point $s \in S$, denote by τ_s the type of the fiber $f_s\colon C_s^\circ \to \mathfrak{P}$ with graph G_s. We define the curve class decoration $\boldsymbol{\beta}_s$ of f_s such that for each $V \in \mathbf{V}(G_s)$, $\boldsymbol{\beta}_s(V) = (\mathfrak{t} \circ f_s)_*[Z_V]$ where $Z_V \subset C_s^\circ$ is the irreducible component corresponding to V.

We say that f is *marked (resp. weakly marked) by a decorated type* $\boldsymbol{\tau} = (\tau, \overline{\gamma}, \boldsymbol{\beta})$ as in (2.3.1) if

1. $\mathfrak{f}$ is marked by τ, see §B.3.3.
2. For each $h \in \mathbf{H}(G)$, the corresponding gerbe $p_h \to \mathfrak{P}_{\mathbf{k}}$ over S is given by $\overline{\gamma}_h$.
3. For each geometric point $s \in S$ with the tropical type τ_s, the contraction $\phi_s\colon \tau_s \to \tau$ of types §B.3.2 is compatible with curve class decoration in the sense that
$$\boldsymbol{\beta}(V) = \sum_{V' \in \mathbf{V}(\phi_s)^{-1}(V)} \boldsymbol{\beta}_s(V')$$
for any $V \in \mathbf{V}(G)$.

2.3.3. *The balancing condition.* For a punctured R-map $f\colon C^\circ \to \mathfrak{P}$ over a geometric log point S, the geometric balancing condition (B.21) translates to

$$f^*\mathcal{O}(\infty_{\mathfrak{P}})|_{Z_V} \cong \mathcal{O}_{Z_V}\Big(\sum_{h \in \mathbf{H}(V)} \mathbf{c}(h) p_h\Big) \tag{2.19}$$

for any $V \in \mathbf{V}(G)$ with $Z_V \subset C^\circ$ the corresponding irreducible component.

Define $\mathbf{deg}(V) := \deg f^*\mathcal{O}(\infty_{\mathfrak{P}})|_{Z_V}$. Taking degrees on both sides of (2.19), we obtain the *balancing condition*

$$\mathbf{deg}(V) = \sum_{h \in \mathbf{H}(G)} \mathbf{c}(h)\mathbf{deg}(h), \tag{2.20}$$

which is compatible with the vertex degree assignment (B.20).

2.4. **Moduli of stable punctured R-maps.** For a decorated type $\boldsymbol{\tau} = (\tau, \overline{\gamma}, \boldsymbol{\beta})$, consider the categories $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau})$ fibered over fs log schemes, and parameterizing punctured R-maps to $\mathfrak{P}$ marked by τ respectively. The goal of this subsection is to prove the following:

Theorem 2.5. *The category $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau})$ is represented by a proper Deligne–Mumford stack with the basic log structure of its universal stable punctured R-map.*

The proof is similar to the log R-map case in [18, Thm. 2.14], and is divided into the following steps.

2.4.1. *Representability.* We verify that $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau})$ is represented by a log Deligne-Mumford stack locally of finite type. Consider the stack $\mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega)$ of underlying R-maps with $\underline{f}$ representable, which associated to any scheme $\underline{S}$ the category of commutative triangles

$$\begin{tikzcd}
 & \underline{\mathfrak{P}} \arrow[d] \\
\underline{C} \arrow[ur, "\underline{f}"] \arrow[r] & \mathbf{BC}^*_\omega
\end{tikzcd} \tag{2.21}$$

where $\underline{C} \to \underline{S}$ is a family of pre-stable curves. Let $\underline{C} \to \underline{C}^c$ be the coarse moduli so that $\underline{C}^c \to \underline{S}$ is the corresponding coarse pre-stable curve. Since the bottom arrow $\underline{C} \to \mathbf{BC}^*_\omega$ factors through $[\omega^{\log}_{\underline{C}^c/\underline{S}}] \colon \underline{C}^c \longrightarrow \mathbf{BC}^*_\omega$, Diagram (2.21) is equivalent to

$$\begin{tikzcd}
 & \underline{\mathfrak{P}}_{\underline{C}^c} \arrow[d] \\
\underline{C} \arrow[ur, "\underline{f}_{\underline{C}^c}"] \arrow[r] & \underline{C}^c
\end{tikzcd} \tag{2.22}$$

where $\underline{\mathfrak{P}}_{\underline{C}^c} = \underline{\mathfrak{P}} \times_{\mathbf{BC}^*_\omega} \underline{C}^c$. Since $\underline{f}$ is representable, $\underline{f}_{\underline{C}^c}$ is a usual stable map over $\underline{S}$ with target $\underline{\mathfrak{P}}_{\underline{C}^c} \to \underline{S}$ a proper, flat family of Deligne-Mumford stacks with projective coarse fibers.

As shown in [18, §5.1], the stack $\mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega)$ parameterizing stable maps $\underline{f}_{\underline{C}^c}$ making (2.22) commutative, is represented by an algebraic stack locally of finite type. Let $\mathfrak{M}(\underline{\mathcal{A}})$ be the stack of usual pre-stable maps to $\underline{\mathcal{A}}$. Define $\mathfrak{R}(\mathfrak{P}, \tau)$ via the following Cartesian diagram of log stacks

$$\begin{tikzcd}
\mathfrak{R}(\mathfrak{P}, \tau) \arrow[r] \arrow[d] & \mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega) \arrow[d] \\
\mathfrak{M}(\mathcal{A}, \tau) \arrow[r] & \mathfrak{M}(\underline{\mathcal{A}})
\end{tikzcd}$$

where we equip both $\mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega)$ and $\mathfrak{M}(\underline{\mathcal{A}})$ with their canonical log structure from the domain curves, the right vertical arrow is induced by the underlying of (2.5), and the bottom arrow is induced by the canonical morphism $\mathcal{A} \to \underline{\mathcal{A}}$. Thus $\mathfrak{R}(\mathfrak{P}, \tau)$ parameterizes pre-stable punctured R-maps whose associated puncture maps to $\mathcal{A}$ are marked by τ. By construction $\mathfrak{R}(\mathfrak{P}, \tau)$ is a log algebraic stack locally of finite type.

We next construct a chain of Deligne-Mumford sub-stacks in $\mathfrak{R}(\mathfrak{P}, \tau)$

$$\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}) \subset \mathscr{R}(\mathfrak{P}, \tau, \boldsymbol{\beta}) \subset \mathscr{R}(\mathfrak{P}, \tau) \subset \mathfrak{R}(\mathfrak{P}, \tau). \tag{2.23}$$

As an open condition, the stability §2.1.4 selects the open substack $\mathscr{R}(\mathfrak{P}, \tau) \subset \mathfrak{R}(\mathfrak{P}, \tau)$ parameterizing stable punctured R-maps. We observe that $\mathscr{R}(\mathfrak{P}, \tau)$ is already a Deligne-Mumford stack. Indeed, consider a stable punctured R-map $f \colon C^\circ \to \mathfrak{P}$ over a geometric log point S. By [18, Prop. 5.1], the automorphism group of the underlying R-map $\underline{f}$ is finite. Lemma B.16 then implies that the automorphism group $\mathrm{Aut}_S(f)$ is also finite.

Next, the curve class assignment $\boldsymbol{\beta}(V)$ for each $V \in \mathbf{L}(G)$ is a discrete data, hence selects an open and closed substack $\mathscr{R}(\mathfrak{P}, \tau, \boldsymbol{\beta}) \subset \mathscr{R}(\mathfrak{P}, \tau)$ consisting of punctured R-maps compatible with the curve class decoration $\boldsymbol{\beta}$.

Finally, the sector decoration $\overline{\boldsymbol{\gamma}}$ further cuts out the strict closed substack $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}) \subset \mathscr{R}(\mathfrak{P}, \tau, \boldsymbol{\beta})$ as follows. Indeed, for any $h \in \mathbf{H}(G)$ with $\mathbf{c}(\mathbf{h}) \neq \mathbf{0}$ (resp. $\mathbf{c}(\mathbf{h}) = \mathbf{0}$), if the assignment $\overline{\gamma}_h$ is an ∞-sector (resp. $\mathfrak{P}_{\mathbf{k}}$-sector), this is a discrete invariants that selects an open and closed substack of $\mathscr{R}(\mathfrak{P}, \tau, \boldsymbol{\beta})$. For any $h \in \mathbf{L}(G) \cup \mathbf{E}(G)$ with $\mathbf{c}(\mathbf{h}) = \mathbf{0}$, if the assignment $\overline{\gamma}_x$ is an ∞-sector or a 0-sector, this is a closed condition that further cuts out a strict closed substack of $\mathscr{R}(\mathfrak{P},\tau,\boldsymbol{\beta})$. Therefore $\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})$ is a log Deligne-Mumford stack locally of finite type.

2.4.2. *Boundedness.* Let $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ be the decorated graph of τ. Consider the following composition of tautological morphisms

$$\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}) \to \mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega) \to \mathfrak{M} := \mathfrak{M}(G,\mathbf{g},\mathbf{m}), \tag{2.24}$$

where $\mathfrak{M}$ is the moduli of untwisted pre-stable curves marked by the decorated graph $(G,\mathbf{g},\mathbf{m})$, and the arrow to $\mathfrak{M}$ is given by taking the coarse domain curve.

First recall from [18, §5.3.2] that the image of the composition (2.24) is contained in a finite type substack $T \subset \mathfrak{M}$. More explicitly, consider any underlying R-map $\underline{f}\colon \underline{C} \to \underline{\mathfrak{P}}$ over a geometric point $\underline{S}$ with curve class $\beta \in H_2(\mathcal{X})$. It was shown in [18, §5.3.2] that for a fixed β, if $\underline{f}$ satisfies the stability condition (2.1.4), then the number of irreducible components of $\underline{C}$ is bounded by a positive integer determined by β. Thus (2.24) becomes

$$\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}) \to \mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega) \to T. \tag{2.25}$$

It remains to show that the composition (2.25) is of finite type. Analogous to [18, §5.3.1], we show that the morphism $\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}) \to \mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega)$ factors through a finite type substack $\mathfrak{S}_T(\tilde{\beta}) \subset \mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega)$ parameterizing possible underlying R-maps satisfying the constraints from $\boldsymbol{\beta}$ and $\mathbf{c}$. Further applying Proposition B.31 and arguing as in [18, §5.3.1], one shows that $\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}) \to \mathfrak{S}_T(\tilde{\beta})$ is of finite type, if $\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})$ consists of finitely many tropical types. As the possible underlying R-maps $\mathfrak{S}_T(\tilde{\beta})$ are of finite type, it suffices to show that the choices of possible contact orders at nodes are finite. This may be proven analogously to [20, Lemma 4.1.5] by inductively applying the balancing condition (2.20).

2.4.3. *Valuative criterion.* The proof of the valuative criterion is analogous to [18, §5.4]. We explain how the same proof may be applied to the situation of punctured R-maps. To set-up the problem, let R be a discrete valuation ring with the quotient field K and the maximal ideal $\mathfrak{m}$. Consider a commutative diagram of solid arrows of underlying stacks

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "{[f_K]}"] \arrow[d] & \underline{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})} \arrow[r] & \mathfrak{S}(\underline{\mathfrak{P}}/\mathbf{BC}^*_\omega) \arrow[d] \\
\operatorname{Spec} R \arrow[rr] \arrow[ur, dashed, "{[f]}"] \arrow[urr, dotted, "{[\underline{f}]}"'] & & \operatorname{Spec} \mathbf{k}
\end{tikzcd}$$

where the top arrow is defined by a basic stable punctured R-map $f_K\colon C^\circ_K \to \mathfrak{P}$ over η with the underlying $\underline{\eta} = \operatorname{Spec} K$. Then the criterion states that, possibly after replacing R by a finite extension of DVRs, and K be the induced finite field extension, there exists a unique dashed arrow $[f]$, represented by a basic stable punctured R-map $f\colon C^\circ \to \mathfrak{P}$ over S with $\underline{S} = \operatorname{Spec} R$, making the above diagram commutative.

Similar to [20, Prop. 2.17], applying Proposition B.32, one observes that the uniqueness and existence of $[f]$ follow from the uniqueness and existence of underlying R-map $\underline{f}\colon \underline{C} \to \underline{\mathfrak{P}}$ over $\underline{S}$ satisfying the stability §2.1.4, leading to the dotted arrow $[\underline{f}]$.

The uniqueness of $\underline{f}$ is verified in [18, §5.5], and the existence of $\underline{f}$ is proved in [18, §5.6, 5.7, 5.8]. Viewing $\underline{f}$ as usual stable maps as in (2.22), the key is to combine the valuative criterion of usual stable maps with the stability §2.1.4.

This completes the proof of Theorem 2.5. ☐

2.5. **Logarithmic evaluation stacks.** We introduce logarithmic enhancements of the evaluations in §2.2 which will be crucial in the construction of the reduced theory §4, and the gluing in §5. We fix a type $\tau = (\mathbf{G},\boldsymbol{\sigma},\mathbf{c})$ as in (B.19) with $\mathbf{G} = (G,\mathbf{g},\mathbf{deg},\mathbf{m})$.

2.5.1. *Logarithmic evaluation along a half-edge.* For each $h \in \mathbf{H}(G)$, consider the stack $\mathfrak{M}^h(\mathcal{A},\tau)$ which associates to any fs log scheme S the category of commutative diagrams

$$\begin{tikzcd} p_h \arrow[r, "ev_h"] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\ C^\circ \arrow[r, "\mathfrak{f}"] & \mathcal{A} \end{tikzcd} \tag{2.26}$$

where $\mathfrak{f}$ is a punctured map over S marked by τ, $p_h \to C^\circ$ is the gerbe corresponding to h as in Notation B.2, and ev_h is representable. Similarly, we define the category $\mathfrak{M}'^{,h}(\mathcal{A},\tau)$ by weaken the τ-marking of $\mathfrak{f}$ in (2.26) by weak τ-marking.

Lemma 2.6. *(1) The stack $\mathfrak{M}^h(\mathcal{A},\tau)$ is an fs log algebraic stack.*

(2) The morphism $\mathfrak{M}^h(\mathcal{A},\tau) \to \mathfrak{M}(\mathcal{A},\tau)$ is strict, locally of finite type, and of Deligne–Mumford type.

Furthermore, analogous statements as above hold for $\mathfrak{M}'^{,h}(\mathcal{A},\tau)$ and $\mathfrak{M}'^{,h}(\mathcal{A},\tau) \to \mathfrak{M}'(\mathcal{A},\tau)$ in the case of weak markings.

Proof. Denote by $\mathfrak{f}_\tau \colon C^\circ_\tau \to \mathcal{A}$ the universal punctured map over $\mathfrak{M}(\mathcal{A},\tau)$, and denote by $p_{\tau,h} \subset C^\circ_\tau$ the strict closed substack corresponding to h. We introduce two stacks

$$\mathrm{Hom}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathcal{A}}\right), \qquad \mathrm{Hom}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathfrak{P}_{\mathbf{k}}}\right)$$

where the stack on the left associates to each $\underline{\mathfrak{M}(\mathcal{A},\tau)}$-scheme $\underline{S}$ a category of morphisms $\mathfrak{f}_h \colon \underline{p_{\tau,h}} \times_{\underline{\mathfrak{M}(\mathcal{A},\tau)}} \underline{S} \to \underline{\mathcal{A}}$, and the stack on the right associates to each $\underline{\mathfrak{M}(\mathcal{A},\tau)}$-scheme $\underline{S}$ a morphism $ev_h \colon \underline{p_{\tau,h}} \times_{\underline{\mathfrak{M}(\mathcal{A},\tau)}} \underline{S} \to \underline{\mathfrak{P}_{\mathbf{k}}}$. By [26, Thm. 1.2], both stacks are algebraic, and their projections to $\underline{\mathfrak{M}(\mathcal{A},\tau)}$ are locally of finite type, quasi-separated, and having affine stabilizers. We further take the open substack consisting of representable morphisms

$$\mathrm{Hom}^{rep}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathfrak{P}_{\mathbf{k}}}\right) \subset \mathrm{Hom}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathfrak{P}_{\mathbf{k}}}\right).$$

Now consider the tautological morphisms

$$\mathfrak{M}(\mathcal{A},\tau) \to \mathrm{Hom}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathcal{A}}\right), \qquad \mathrm{Hom}^{rep}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathfrak{P}_{\mathbf{k}}}\right) \to \mathrm{Hom}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathcal{A}}\right),$$

where the first one is obtained by restriction to $p_{\tau,h}$, and the second morphism is obtained by composing with $\underline{\mathfrak{P}_{\mathbf{k}}} \to \underline{\mathcal{A}}$. By the definition of $\mathfrak{M}^h(\mathcal{A},\tau)$, observe that

$$\mathfrak{M}^h(\mathcal{A},\tau) \cong \mathfrak{M}(\mathcal{A},\tau) \times_{\mathrm{Hom}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathcal{A}}\right)} \mathrm{Hom}^{rep}_{\underline{\mathfrak{M}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}}, \underline{\mathfrak{P}_{\mathbf{k}}}\right)$$

where the log structure on $\mathfrak{M}^h(\mathcal{A},\tau)$ is obtained by pulling back from $\mathfrak{M}(\mathcal{A},\tau)$. This shows the algebraicity of $\mathfrak{M}^h(\mathcal{A},\tau)$. Further observe that the projection $\mathfrak{M}^h(\mathcal{A},\tau) \to \mathfrak{M}(\mathcal{A},\tau)$ given by the product construction is strict and locally of finite type.

Finally, to see that $\mathfrak{M}^h(\mathcal{A},\tau) \to \mathfrak{M}(\mathcal{A},\tau)$ is of Deligne-Mumford type, it suffices to observe that the automorphisms of (2.26) fixing $\mathfrak{f}$ is necessarily finite over S as $\mathfrak{P}_{\mathbf{k}}$ is of Deligne-Mumford type. This finishes the proof of (1) and (2).

The case of $\mathfrak{M}'^{,h}(\mathcal{A},\tau)$ is similar, and is omitted. □

For later use, denote the universal object over $\mathfrak{M}^h(\mathcal{A},\tau)$ by

$$\begin{tikzcd} p_{\tau,h} \arrow[r, "ev_h"] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\ C^\circ_\tau \arrow[r, "\mathfrak{f}_\tau"] & \mathcal{A} \end{tikzcd} \tag{2.27}$$

with the projection $\pi_h \colon p_{\tau,h} \to \mathfrak{M}^h(\mathcal{A},\tau)$. To study the smoothness and decomposition of $\mathfrak{M}^h(\mathcal{A},\tau)$, we further impose:

Assumption 2.7. $\mathcal{X}$ is smooth.

In particular, this assumption implies that $\mathfrak{P} \to \mathbf{BC}^*_\omega$ is log smooth with the smooth underlying morphism, hence the morphism has a log tangent bundle $T_{\mathfrak{P}/\mathbf{BC}^*_\omega}$. Taking the base change $\operatorname{Spec}\mathbf{k} \to \mathbf{BC}^*_\omega$, we obtain the natural strict and smooth morphism $\mathfrak{P}_{\mathbf{k}} \to \mathcal{A}$. In particular the log and underlying tangent bundles agree $T_{\mathfrak{P}_{\mathbf{k}}/\mathcal{A}} \cong T_{\underline{\mathfrak{P}}_{\mathbf{k}}/\underline{\mathcal{A}}}$. Restricting to $\infty_{\mathfrak{P}_{\mathbf{k}}} \to \infty_{\mathcal{A}}$ and $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}} \to \operatorname{Spec}\mathbf{k} \in \mathcal{A}$, we obtain the corresponding tangent bundles

$$T_{\mathfrak{P}_{\mathbf{k}}/\mathcal{A}}|_{\infty_{\mathfrak{P}_{\mathbf{k}}}} \cong T_{\infty_{\mathfrak{P}_{\mathbf{k}}}/\infty_{\mathcal{A}}} \cong T_{\underline{\infty_{\mathfrak{P}_{\mathbf{k}}}}/\underline{\infty_{\mathcal{A}}}}, \qquad T_{\mathfrak{P}_{\mathbf{k}}/\mathcal{A}}|_{\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}} \cong T_{\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}}.$$

2.5.2. *Decomposition and smoothness of $\mathfrak{M}^h(\mathcal{A},\tau)$ for $\mathbf{c}(h) \neq 0$.* In this case, the map ev_h in (2.26) factors through $\infty_{\mathfrak{P}_{\mathbf{k}}} \subset \mathfrak{P}_{\mathbf{k}}$. By §2.2.3, we have open and closed decompositions

$$\mathfrak{M}^h(\mathcal{A},\tau) = \bigsqcup_{\overline{\gamma}_h \subset \overline{\mathcal{I}}_\mu \infty_{\mathfrak{P}_{\mathbf{k}}}} \mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau), \qquad \mathfrak{M}'^{,h}(\mathcal{A},\tau) = \bigsqcup_{\overline{\gamma}_h \subset \overline{\mathcal{I}}_\mu \infty_{\mathfrak{P}_{\mathbf{k}}}} \mathfrak{M}'^{,\overline{\gamma}_h}(\mathcal{A},\tau) \tag{2.28}$$

where both unions run through all ∞-sectors, and $\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau) \subset \mathfrak{M}^h(\mathcal{A},\tau)$ and $\mathfrak{M}'^{,\overline{\gamma}_h}(\mathcal{A},\tau) \subset \mathfrak{M}'^{,h}(\mathcal{A},\tau)$ are the component over which ev_h in (2.27) factors through the universal gerbe $\gamma_h \to \infty_{\mathfrak{P}_{\mathbf{k}}}$ over $\overline{\gamma}_h$.

Lemma 2.8. *Suppose that $\mathbf{c}(h) \neq 0$, and Assumption (2.7) holds. Then, the strict morphism $\mathfrak{M}^h(\mathcal{A},\tau) \to \mathfrak{M}(\mathcal{A},\tau)$ is smooth with relative tangent bundle*

$$T_{\underline{\mathfrak{M}^h(\mathcal{A},\tau)}/\underline{\mathfrak{M}(\mathcal{A},\tau)}} \cong T_{\mathfrak{M}^h(\mathcal{A},\tau)/\mathfrak{M}(\mathcal{A},\tau)} \cong \pi_{h,*}\mathit{ev}_h^* T_{\mathfrak{P}_{\mathbf{k}}/\mathcal{A}} \cong \pi_{h,*}\mathit{ev}_h^* T_{\infty_{\mathfrak{P}_{\mathbf{k}}}/\infty_{\mathcal{A}}}. \tag{2.29}$$

The same statement holds for $\mathfrak{M}'^{,h}(\mathcal{A},\tau) \to \mathfrak{M}'(\mathcal{A},\tau)$

Proof. It suffices to prove formal smoothness. Let $T_0 \to T$ be a strict closed embedding of affine schemes defined by an ideal I satisfying $I^2 = 0$. Consider a commutative diagram of solid arrows

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d] & \mathfrak{M}^h(\mathcal{A},\tau) \arrow[d] \\
T \arrow[r, "{[\mathfrak{f}_T]}"'] \arrow[ur, dashed] & \mathfrak{M}(\mathcal{A},\tau)
\end{tikzcd} \tag{2.30}$$

We want to show that there exists a dashed arrow making the above diagram commutative. This amounts to showing the existence of a dashed arrow making the following diagram commutative

$$\begin{tikzcd}
p_{h,T_0} \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathit{ev}_{h,T_0}"] & p_{h,T} \arrow[r, dashed, "\mathit{ev}_{h,T}"'] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\
C^\circ_{T_0} \arrow[r] \arrow[d] & C^\circ_T \arrow[r, "\mathfrak{f}_T"'] \arrow[d] & \mathcal{A} \\
T_0 \arrow[r] & T &
\end{tikzcd} \tag{2.31}$$

where the two left squares are Cartesian. As the morphisms ev_{h,T_0} and $\mathit{ev}_{h,T}$ necessarily factor through $\infty_{\mathfrak{P}_{\mathbf{k}}}$, it suffices to understand the dashed arrow $\mathit{ev}_{h,T}$ making the following square commutative

$$\begin{tikzcd}
p_{h,T_0} \arrow[r, "\mathit{ev}_{h,T_0}"] \arrow[d] & \infty_{\mathfrak{P}_{\mathbf{k}}} \arrow[d] \\
p_{h,T} \arrow[r] \arrow[ur, dashed] & \infty_{\mathcal{A}}
\end{tikzcd}$$

Since $p_{h,T_0} \to p_{h,T}$ is also a square-zero extension defined by I, by the smoothness of $\infty_{\mathfrak{P}_{\mathbf{k}}} \to \infty_{\mathcal{A}}$, the lifting $\mathit{ev}_{h,T}$ exists, and all liftings are classified by

$$(\pi_{h,T_0})_*(\mathit{ev}_{h,T_0})^* T_{\infty_{\mathfrak{P}_{\mathbf{k}}}/\infty_{\mathcal{A}}} \cong (\pi_{h,T_0})_*(\mathit{ev}_{h,T_0})^* T_{\mathfrak{P}_{\mathbf{k}}/\mathcal{A}},$$

where $\pi_{h,T_0} \colon p_{h,T_0} \to T_0$ is the projection. This finishes the proof in the case of τ-markings. The case of weak τ-markings is identical and is omitted. $\square$

2.5.3. *Decomposition and smoothness of* $\mathfrak{M}^h(\mathcal{A},\tau)$ *for* $\mathbf{c}(h) = 0$. In this case, consider the open substack $\mathfrak{M}(\tau, \mathring{h}) \subset \mathfrak{M}(\mathcal{A},\tau)$ parameterizing punctured maps such that the image of the gerbe corresponding to h avoids $\infty_{\mathcal{A}}$, hence the corresponding open substack

$$\mathfrak{M}^{\mathring{h}}(\mathcal{A},\tau) := \mathfrak{M}^h(\mathcal{A},\tau) \times_{\mathfrak{M}(\mathcal{A},\tau)} \mathfrak{M}(\tau,\mathring{h}) \subset \mathfrak{M}^h(\mathcal{A},\tau)$$

with the universal object

$$\begin{tikzcd} p_{\tau,\mathring{h}} \arrow[r, "\mathit{ev}_{\mathring{h}}"] \arrow[d] & \mathfrak{P}^\circ_{\mathbf{k}} \arrow[d] \\ C^\circ_\tau \arrow[r, "\mathfrak{f}_\tau"] & \mathcal{A} \end{tikzcd} \tag{2.32}$$

and the projection $\pi_{\mathring{h}} \colon p_{\tau,\mathring{h}} \to \mathfrak{M}^{\mathring{h}}(\mathcal{A},\tau)$. Recall that $\mathfrak{P}^\circ_{\mathbf{k}} = \mathfrak{P}_{\mathbf{k}} \setminus \infty_{\mathfrak{P}_{\mathbf{k}}} \to \mathcal{A}$ factors through the open substack $\operatorname{Spec} \mathbf{k} \subset \mathcal{A}$.

Lemma 2.9. *Suppose that* $h \in \mathbf{H}(G)$, $\mathbf{c}(h) = 0$, *and Assumption* (2.7) *holds. The strict morphism* $\mathfrak{M}^{\mathring{h}}(\mathcal{A},\tau) \to \mathfrak{M}(\tau,\mathring{h})$ *is smooth with the tangent bundle*

$$T_{\mathfrak{M}^{\mathring{h}}(\mathcal{A},\tau)/\mathfrak{M}(\tau,\mathring{h})} \cong \pi_{\mathring{h},*} \mathit{ev}^*_{\mathring{h}} T_{\mathfrak{P}^\circ_{\mathbf{k}}}.$$

Proof. This is similar to the proof of Lemma 2.9 with $\mathfrak{P}_{\mathbf{k}}$ replaced by $\mathfrak{P}^\circ_{\mathbf{k}}$. We omit the details here. $\square$

For each $h \in \mathbf{H}(G)$ with $\mathbf{c}(h) = 0$, similar to (2.28) we have the following open and closed decomposition

$$\mathfrak{M}^{\mathring{h}}(\mathcal{A},\tau) = \coprod_{\overline{\gamma}_{\mathring{h}} \subset \overline{\mathcal{I}}_\mu \mathfrak{P}_{\mathbf{k}}} \mathfrak{M}^{\overline{\gamma}_{\mathring{h}}}(\mathcal{A},\tau) \tag{2.33}$$

where the union runs through all connected components $\overline{\gamma}_{\mathring{h}} \subset \overline{\mathcal{I}}_\mu \mathfrak{P}^\circ_{\mathbf{k}}$, and $\mathfrak{M}^{\overline{\gamma}_{\mathring{h}}}(\mathcal{A},\tau)$ is the component over which the evaluation $\mathit{ev}_{\mathring{h}}$ in (2.32) factors through the universal gerbe $\gamma_{\mathring{h}} \to \mathfrak{P}^\circ_{\mathbf{k}}$ over $\overline{\gamma}_{\mathring{h}}$. The restriction of $\mathit{ev}_{\mathring{h}}$ to each component $\mathfrak{M}^{\overline{\gamma}_{\mathring{h}}}(\mathcal{A},\tau)$ induces a canonical morphism

$$\mathrm{ev}_h \colon \mathfrak{M}^{\overline{\gamma}_{\mathring{h}}}(\mathcal{A},\tau) \to \overline{\gamma}_{\mathring{h}}.$$

By Lemma 2.9 and [5, Lemma 3.6.1], we have that

$$T_{\mathfrak{M}^{\overline{\gamma}_{\mathring{h}}}(\mathcal{A},\tau)/\mathfrak{M}(\tau,\mathring{h})} \cong \mathrm{ev}^*_h T_{\overline{\gamma}_{\mathring{h}}}. \tag{2.34}$$

2.5.4. *Half-edges with* 0*-sectors.* We continue to assume that $\mathbf{c}(h) = 0$, and consider the strict closed substack $\mathfrak{M}^{h^{\mathrm{cpt}}}(\mathcal{A},\tau) \subset \mathfrak{M}^{\mathring{h}}(\mathcal{A},\tau)$ such that the image of the gerbe corresponding to h factors through $\mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$. We have a decomposition to open and closed substacks

$$\mathfrak{M}^{h^{\mathrm{cpt}}}(\mathcal{A},\tau) = \coprod_{\overline{\gamma}_h \subset \overline{\mathcal{I}}_\mu \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}} \mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)$$

where the union runs through all 0-sectors, and $\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)$ is the component over which the evaluation ev_h factors through the universal gerbe $\gamma_h \to \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ over $\overline{\gamma}_h$.

The component $\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)$ can be also constructed as follows. The universal gerbe $\gamma_h \to \mathbf{0}_{\mathfrak{P}_{\mathbf{k}}}$ can be viewed as a connected family of gerbes in $\mathfrak{P}^\circ_{\mathbf{k}}$. Thus there is a unique connected component $\overline{\gamma}_{\mathring{h}} \subset \overline{\mathcal{I}}_\mu \mathfrak{P}^\circ_{\mathbf{k}}$ and a closed embedding $\overline{\gamma}_h \to \overline{\gamma}_{\mathring{h}}$ such that the universal gerbe over $\overline{\gamma}_h$ is obtained by pulling back the universal gerbe over $\overline{\gamma}_{\mathring{h}}$. We thus obtain a Cartesian diagram

$$\begin{tikzcd}
\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau) \arrow[r] \arrow[d] & \mathfrak{M}^{\overline{\gamma}_{\mathring{h}}}(\mathcal{A},\tau) \arrow[d, "\mathrm{ev}_h"] \\
\overline{\gamma}_h \arrow[r] & \overline{\gamma}_{\mathring{h}}
\end{tikzcd}$$

Pulling back (2.32), we obtain the universal objects over $\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)$

$$\begin{tikzcd}
p_{\tau,\overline{\gamma}_h} \arrow[r, "ev_h"] \arrow[d] & \mathbf{0}_{\mathfrak{P}_\mathbf{k}} \arrow[d] \\
C^\circ_\tau \arrow[r, "\mathfrak{f}_\tau"] & \mathcal{A}
\end{tikzcd} \tag{2.35}$$

with the projection $\pi_h \colon p_{\tau,\overline{\gamma}_h} \to \mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)$, where the top horizontal arrow is a family of gerbes in $\mathbf{0}_{\mathfrak{P}_\mathbf{k}}$ given by $\overline{\gamma}_h$. By Lemma 2.9 and (2.34), we have

$$T_{\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)/\mathfrak{M}(\mathcal{A},\tau)} \cong \pi_{h,*} ev_h^* T_{\mathbf{0}_{\mathfrak{P}_\mathbf{k}}} \cong \mathrm{ev}_h^* T_{\overline{\gamma}_h}. \tag{2.36}$$

Remark 2.10. Similarly for a half-edge h with $\mathbf{c}(h) = 0$, we define the strict closed substack $\mathfrak{M}'^{,\overline{\gamma}_h}(\mathcal{A},\tau) \subset \mathfrak{M}'^{,h}(\mathcal{A},\tau)$ parameterizing diagrams (2.26) such that $\mathfrak{f}$ is weakly marked by τ, and the underlying of ev_h are gerbes in $\mathbf{0}_{\mathfrak{P}_\mathbf{k}}$ given by $\overline{\gamma}_h$.

Remark 2.11. The discussion in this section, especially the proof of Lemma 2.6, fixes an error in our previous work [18]. More specifically, the proof of [18, Proposition 3.4] incorrectly identifies the logarithmic evaluation stack with a direct product, see [18, (3.8)]. In the rest of this remark, we describe the close relationship between the logarithmic evaluation stack to this direct product.

Assume that $\mathbf{c}(h) = 0$ and $\overline{\gamma}_h$ is a 0-sector. Consider the open substack $\mathfrak{M} \subset \mathfrak{M}(\mathcal{A},\tau)$ parameterizing punctured maps such that the image of the gerbe corresponding to h avoids $\infty_{\mathcal{A}}$. We will show that the strict tautological morphism

$$\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau) \to \mathfrak{M} \times \overline{\gamma}_h \tag{2.37}$$

is finite, étale, and surjective. Here, the product $\mathfrak{M} \times \overline{\gamma}_h$ agrees with (a connected component of) [18, (3.8)].

Consider the two $B\mu_{|\gamma_h|}$-gerbes over $\underline{\mathfrak{M}} \times \overline{\gamma}_h$ given by

$$\underline{p_{\tau,h}} \times \overline{\gamma}_h \to \underline{\mathfrak{M}} \times \overline{\gamma}_h, \qquad \underline{\mathfrak{M}} \times \gamma_h \to \underline{\mathfrak{M}} \times \overline{\gamma}_h \tag{2.38}$$

where $\underline{p_{\tau,h}} \to \underline{\mathfrak{M}}$ is the universal gerbe corresponding to $h \in \mathbf{H}(G)$. Note that there is a tautological morphism

$$\underline{\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)} \longrightarrow \mathrm{Isom}_{\underline{\mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A},\tau)}}\left(\underline{p_{\tau,h}} \times \overline{\gamma}_h, \underline{\mathfrak{M}} \times \gamma_h\right) \tag{2.39}$$

where the right hand side parameterizes isomorphisms of the two $B\mu_{|\gamma_h|}$-gerbes. We further observe that this tautological morphism is an isomorphism. Indeed, for any strict morphism $S \to \mathfrak{M} \times \overline{\gamma}_h$ from a log scheme S, denote by $\underline{p_h} \to \underline{S}$ and $(\gamma_h)_{\underline{S}} \to \underline{S}$ the pull-back of the two gerbes in (2.38). If we are further given an isomorphism of gerbes $\underline{p_h} \xrightarrow{\cong} (\gamma_h)_{\underline{S}}$, we obtain a commutative square (2.35) with the arrow ev_h defined by the composition $\underline{p_h} \to (\gamma_h)_{\underline{S}} \to \underline{\gamma_h} \to \underline{\mathbf{0}_{\mathfrak{P}_\mathbf{k}}}$, defining the inverse of (2.39).

Note that smooth-locally over $\underline{\mathfrak{M}} \times \overline{\gamma}_h$, both gerbes in (2.38) are isomorphic to the trivial $B\mu_r$-gerbes for $r_h = |\gamma_h|$. Since the statement is local on $\underline{\mathfrak{M}} \times \overline{\gamma}_h$, consider a smooth morphism $\underline{T} \to \underline{\mathfrak{M}} \times \overline{\gamma}_h$ from a scheme $\underline{T}$, over which (2.38) are trivialized. It suffices to show that

$$\mathrm{Isom}_{\underline{T}}\left(B\mu_{r_h} \times \underline{T}, B\mu_{r_h} \times \underline{T}\right) \to \underline{T}$$

is finite, étale and surjective.

Now we observe that a morphism $g\colon B\mu_{r_h} \times \underline{T} \to B\mu_{r_h} \times \underline{T}$ over $\underline{T}$ being an isomorphism is equivalent to g being representable. This implies that

$$\mathrm{Isom}_{\underline{T}}\left(B\mu_{r_h} \times \underline{T}, B\mu_{r_h} \times \underline{T}\right) \subset \overline{\mathcal{I}}_{\mu} B\mu_{r_h} \times \underline{T}$$

consists of components parameterizing μ_{r_h}-gerbes. Consequently we have

$$\mathrm{Isom}_{\underline{T}}\left(B\mu_{r_h} \times \underline{T}, B\mu_{r_h} \times \underline{T}\right) = \sqcup \underline{T}$$

consisting of $\phi(r_h)$ disjoint copies of $\underline{T}$ with $\phi(r_h)$ the number of integers in $\{1, 2, \cdots, r_h\}$ coprime to r_h. This proves the claim.

2.5.5. *Logarithmic evaluations of edges.* Consider an edge $x = \{h, \hat{h}\} \in \mathbf{E}(G)$. Define the stack $\mathfrak{M}^x(\mathcal{A}, \tau)$ to be the fibered category which associates to any fs log scheme S the category of commutative diagrams of log stacks

$$\begin{tikzcd} p_h \sqcup p_{\hat{h}} \arrow[r, "\mathit{ev}_h \sqcup \mathit{ev}_{\hat{h}}"] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\ C^{\circ} \arrow[r, "\mathfrak{f}"] & \mathcal{A} \end{tikzcd} \tag{2.40}$$

with the two morphisms $\mathit{ev}_h, \mathit{ev}_{\hat{h}}$ satisfying (2.15), i.e. $\mathit{ev}_{\hat{h}} = \iota_{\omega} \circ \mathit{ev}_h \circ \iota_h$. Here $p_h, p_{\hat{h}} \subset C^{\circ}$ are the two gerbes corresponding to the two half-edges over the node $p_x \subset C^{\circ}$. Thus ev_h and $\mathit{ev}_{\hat{h}}$ determine each other. This leads to two isomorphisms

$$\mathfrak{M}^x(\mathcal{A}, \tau) \longrightarrow \mathfrak{M}^h(\mathcal{A}, \tau), \qquad \mathfrak{M}^x(\mathcal{A}, \tau) \longrightarrow \mathfrak{M}^{\hat{h}}(\mathcal{A}, \tau) \tag{2.41}$$

We will be mainly interested in the case of $\mathbf{c}(x) > 0$. In this case, (2.28) implies a decomposition

$$\mathfrak{M}^x(\mathcal{A}, \tau) = \sqcup_{\overline{\gamma}_x} \mathfrak{M}^{\overline{\gamma}_x}(\mathcal{A}, \tau)$$

where $\overline{\gamma}_x = (\overline{\gamma}_h, \overline{\gamma}_{\hat{h}})$ runs through nodal involuted pairs of ∞-sectors. Note that (2.41) restricts to isomorphisms

$$\mathfrak{M}^{\overline{\gamma}_x}(\mathcal{A}, \tau) \longrightarrow \mathfrak{M}^{\overline{\gamma}_h}(\mathcal{A}, \tau), \qquad \mathfrak{M}^{\overline{\gamma}_x}(\mathcal{A}, \tau) \longrightarrow \mathfrak{M}^{\overline{\gamma}_{\hat{h}}}(\mathcal{A}, \tau). \tag{2.42}$$

Remark 2.12. Similarly for an edge $x = \{h, \hat{h}\} \in \mathbf{E}(G)$, we define the log stack $\mathfrak{M}'^{,\overline{\gamma}_x}(\mathcal{A}, \tau)$ parameterizing diagrams (2.40) with weakly marked $\mathfrak{f}$. The same discussion as above implies isomorphisms

$$\mathfrak{M}'^{,\overline{\gamma}_x}(\mathcal{A}, \tau) \longrightarrow \mathfrak{M}'^{,\overline{\gamma}_h}(\mathcal{A}, \tau), \qquad \mathfrak{M}'^{,\overline{\gamma}_x}(\mathcal{A}, \tau) \longrightarrow \mathfrak{M}'^{,\overline{\gamma}_{\hat{h}}}(\mathcal{A}, \tau).$$

analogous to (2.42).

2.5.6. *Logarithmic evaluation morphisms.* Consider a stable log R-map $f\colon C^{\circ} \to \mathfrak{P}$ over S marked by τ. For each $h \in \mathbf{H}(G)$, we obtain a commutative diagram

$$\begin{tikzcd} p_h \arrow[r] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \arrow[dr, bend left] & \\ C^{\circ} \arrow[r, "\mathfrak{f}"] & \mathfrak{P} \arrow[r] & \mathcal{A} \end{tikzcd} \tag{2.43}$$

with $\mathfrak{f}$ the associated punctured map of f, $p_h \subset C^{\circ}$ the gerbe corresponding to h, and $p_h \to \mathfrak{P}_{\mathbf{k}}$ induced by the restriction $f|_{p_h}$ as in (2.12) and (2.14). This induces the *log evaluation morphism* along h to be

$$\mathbf{ev}_h\colon S \to \mathfrak{M}^h(\mathcal{A}, \tau) \tag{2.44}$$

such that the universal family (2.27) pulls back to the outer square of (2.43).

In case $x = \{h, \hat{h}\} \in \mathbf{E}(G)$ is an edge, we obtain a factorization

$$\begin{tikzcd} S \arrow[rr, "\mathbf{ev}_h \times \mathbf{ev}_{\hat{h}}"] \arrow[dr, "\mathbf{ev}_x"'] & & \mathfrak{M}^h(\mathcal{A}, \tau) \times \mathfrak{M}^{\hat{h}}(\mathcal{A}, \tau) \\ & \mathfrak{M}^x(\mathcal{A}, \tau) \arrow[ur] & \end{tikzcd} \tag{2.45}$$

where the right skewed arrow is given by (2.41). Indeed, the commutative square (2.26) pulls back to (2.43) for both h and $\hat{h}$. We call $\mathbf{ev}_x$ the *log evaluation morphism* along x.

2.6. The canonical perfect obstruction theory.

Fix a type $\tau = (\mathbf{G}, \boldsymbol{\sigma}, \mathbf{c})$ as in (B.19) with $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$, and continue to assume Assumption 2.7.

2.6.1. *The canonical theory of a type.*

Write for simplicity

$$\mathscr{R} := \mathscr{R}(\mathfrak{P}, \tau), \qquad \mathfrak{M} := \mathfrak{M}(\mathcal{A}, \tau),$$

where $\mathscr{R}(\mathfrak{P}, \tau)$ is the stack of stable punctured R-maps with the associated maps to $\mathcal{A}$ marked by τ, see (2.23). Similar to [18, §3.1], the strict tautological morphism

$$\mathscr{R} \longrightarrow \mathfrak{M}$$

admits a *canonical perfect obstruction* constructed as follows.

For $\bullet = \mathscr{R}$ or $\mathfrak{M}$, let $\pi_\bullet \colon C^\circ_\bullet \to \bullet$ be the universal punctured curve, and consider the product in the fs category $\mathfrak{P}_\bullet := C^\circ_\bullet \times_{\mathbf{BC}^*_\omega} \mathfrak{P}$. Denote by $f_{\mathscr{R}} \colon C^\circ_{\mathscr{R}} \to \mathfrak{P}$ the universal punctured R-map over $\mathscr{R}$. Let $\rho \colon C^\circ_{\mathscr{R}} \to \mathfrak{P}_{\mathscr{R}}$ be the morphism induced by $f_{\mathscr{R}}$. These arrows fit in a commutative diagram

$$\begin{tikzcd} C^\circ_{\mathscr{R}} \arrow[dr, "\rho"] \arrow[drr, bend left, "="] \arrow[ddr, bend right] & & & \\ & \mathfrak{P}_{\mathscr{R}} \arrow[r] \arrow[d] & C^\circ_{\mathscr{R}} \arrow[r, "\pi_{\mathscr{R}}"] \arrow[d] & \mathscr{R} \arrow[d] \\ & \mathfrak{P}_{\mathfrak{M}} \arrow[r] & C^\circ_{\mathfrak{M}} \arrow[r, "\pi_{\mathscr{R}}"] & \mathfrak{M} \end{tikzcd}$$

Note that all the three vertical arrows are strict, hence the two squares are Cartesian in both the fs and underlying categories. We will use $\mathbb{L}$ to denote the log cotangent complex in the sense of [33], and $\mathbb{T}$ its dual. Applying [33, (1.1.6)] to both the lower and upper triangles, we obtain

$$\mathbb{L}_{\rho_{\mathscr{R}}} \to \rho^* \mathbb{L}_{\mathfrak{P}_{\mathscr{R}}/\mathfrak{P}_{\mathfrak{M}}}[1] \cong \pi^*_{\mathscr{R}} \mathbb{L}_{\mathscr{R}/\mathfrak{M}},$$
$$\mathbb{L}_{\rho_{\mathscr{R}}} \cong \rho^* \mathbb{L}_{\mathfrak{P}_{\mathscr{R}}/C^\circ_{\mathscr{R}}}[1] \cong f^*_{\mathscr{R}} \mathbb{L}_{\mathfrak{P}/\mathbf{BC}^*_\omega},$$

respectively, hence

$$f^*_{\mathscr{R}} \mathbb{L}_{\mathfrak{P}/\mathbf{BC}^*_\omega} \to \pi^*_{\mathscr{R}} \mathbb{L}_{\mathscr{R}/\mathfrak{M}}.$$

Note that the dualizing complex of the underlying morphism $\underline{\pi_{\mathscr{R}}}$ is $\omega_{\underline{C_{\mathscr{R}}}/\underline{\mathscr{R}}}[1]$, where $\omega_{\underline{C_{\mathscr{R}}}/\underline{\mathscr{R}}}$ is the dualizing line bundle of $\underline{\pi_{\mathscr{R}}}$. By the same argument as in [18, §3.1], the above arrow yields a canonical morphism

$$\varphi^\vee_\tau \colon \mathbb{E}^\vee_{\mathscr{R}/\mathfrak{M}} := R\pi_{\mathscr{R},*} \left(f^*_{\mathscr{R}} T^\vee_{\mathfrak{P}/\mathbf{BC}^*_\omega} \otimes \omega_{\underline{C_{\mathscr{R}}}/\underline{\mathscr{R}}}[1] \right) \longrightarrow \mathbb{L}_{\mathscr{R}/\mathfrak{M}}. \tag{2.46}$$

Taking the dual of (2.46), we obtain

$$\varphi_\tau \colon \mathbb{T}_{\mathscr{R}/\mathfrak{M}} \longrightarrow \mathbb{E}_{\mathscr{R}/\mathfrak{M}} \cong R\pi_{\mathscr{R},*} \left(f^*_{\mathscr{R}} T_{\mathfrak{P}/\mathbf{BC}^*_\omega} \right). \tag{2.47}$$

Similar to [20, Prop. 5.2], one checks using [3, Prop. 4.2] that (2.46) is a perfect obstruction theory for $\mathscr{R} \to \mathfrak{M}$ in the sense of Behrend–Fantechi [8, Def. 4.4]. By the functoriality of [3, Lemma 4.1], (2.47) pulls back to a perfect obstruction theory

$$\varphi_{\mathcal{T}}\colon \mathbb{T}_{\mathscr{R}/\mathfrak{M}} \longrightarrow \mathbb{E}_{\mathscr{R}/\mathfrak{M}} \cong R\pi_{\mathscr{R},*}\left(f^*_{\mathscr{R}} T_{\mathfrak{P}/\mathbf{BC}^*_\omega}\right). \tag{2.48}$$

We will refer to (2.46), or equivalent its dual (2.48) as the *canonical perfect obstruction theory.*

2.6.2. *The canonical theory relative to compact type markings.* Let $\mathbf{L}_0(G) \subset \mathbf{L}(G)$ be the subset of legs with the zero contact order. Consider the products taken over $\mathfrak{M}$:

$$\mathfrak{M}^{\mathrm{cpt}} := \prod_{x \in \mathbf{L}_0(G)} \mathfrak{M}^{x^{\mathrm{cpt}}}(\mathcal{A}, \tau), \quad \mathfrak{M}^{\mathring{\mathbf{L}}_0} := \prod_{x \in \mathbf{L}_0(G)} \mathfrak{M}^{\mathring{x}}(\mathcal{A}, \tau), \quad \mathfrak{M}^{\mathbf{L}_0} := \prod_{x \in \mathbf{L}_0(G)} \mathfrak{M}^{x}(\mathcal{A}, \tau).$$

We obtain tautological strict embeddings

$$\mathfrak{M}^{\mathrm{cpt}} \hookrightarrow \mathfrak{M}^{\mathring{\mathbf{L}}_0} \hookrightarrow \mathfrak{M}^{\mathbf{L}_0}$$

where the right embedding is strict and open, and the left embedding is strict and closed. All three stacks are strict, smooth and of Deligne-Mumford type over $\mathfrak{M}$.

Define stacks $\mathscr{R}^{\mathrm{cpt}}$ and $\mathscr{R}^{\mathring{\mathbf{L}}_0}$ via the following cartesian diagram with strict arrows

$$\begin{tikzcd}
\mathscr{R}^{\mathrm{cpt}} \arrow[r] \arrow[d] & \mathscr{R}^{\mathring{\mathbf{L}}_0} \arrow[r] \arrow[d] & \mathscr{R} \arrow[d, "\mathbf{ev}_{\mathbf{L}_0}"] \\
\mathfrak{M}^{\mathrm{cpt}} \arrow[r] & \mathfrak{M}^{\mathring{\mathbf{L}}_0} \arrow[r] & \mathfrak{M}^{\mathbf{L}_0}
\end{tikzcd} \tag{2.49}$$

where $\mathbf{ev}_{\mathbf{L}_0} = \prod_{h \in \mathbf{L}_0} \mathbf{ev}_h$ is the product of log evaluations of (2.44). Then $\mathscr{R}'^{,\mathring{\mathbf{L}}_0} \subset \mathscr{R}$ is the open substack such that for any $h \in \mathbf{L}_0(G)$ the images of gerbes corresponding to h avoid $\infty_{\mathfrak{P}}$, and $\mathscr{R}^{\mathrm{cpt}} \subset \mathscr{R}^{\mathring{\mathbf{L}}_0}$ is the strict closed substack with images of gerbes corresponding to any $h \in \mathbf{L}_0(G)$ landing in $\mathbf{0}_{\mathfrak{P}}$.

Consider the universal family

$$f_{\mathring{\mathbf{L}}_0}\colon C^{\circ}_{\mathring{\mathbf{L}}_0} \to \mathfrak{P}, \qquad \pi_{\mathring{\mathbf{L}}_0}\colon C^{\circ}_{\mathring{\mathbf{L}}_0} \to \mathscr{R}^{\mathring{\mathbf{L}}_0}.$$

over $\mathscr{R}^{\mathring{\mathbf{L}}_0}$. For each $x \in \mathbf{L}_0(G)$, denote by $p_x \subset C^{\circ}_{\mathring{\mathbf{L}}_0}$ the corresponding gerbe, and write $\Sigma_0 = \sum_{x \in \mathbf{L}_0(G)} p_x$. Define the complex

$$\mathbb{E}_{\mathscr{R}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}^{\mathring{\mathbf{L}}_0}} := R\pi_{\mathring{\mathbf{L}}_0,*}\left(f^*_{\mathring{\mathbf{L}}_0} T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0)\right).$$

By the standard construction (see for example [3, §4.2]), we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{R}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}^{\mathring{\mathbf{L}}_0}} \arrow[r] \arrow[d, "\varphi_{\tau,\mathring{\mathbf{L}}_0}"'] & \mathbb{T}_{\mathscr{R}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}} \arrow[r] \arrow[d, "\varphi_{\tau}"'] & T_{\mathfrak{M}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}}\big|_{\mathscr{R}^{\mathring{\mathbf{L}}_0}} \arrow[r, "{[1]}"] \arrow[d, equal] & {} \\
\mathbb{E}_{\mathscr{R}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}^{\mathring{\mathbf{L}}_0}} \arrow[r] & \mathbb{E}_{\mathscr{R}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}} \arrow[r] & T_{\mathfrak{M}^{\mathring{\mathbf{L}}_0}/\mathfrak{M}}\big|_{\mathscr{R}^{\mathring{\mathbf{L}}_0}} \arrow[r, "{[1]}"] & {}
\end{tikzcd} \tag{2.50}$$

where φ_τ is the canonical perfect obstruction theory obtained by restricting (2.47) to the open substack $\mathscr{R}^{\mathring{\mathbf{L}}_0} \subset \mathscr{R}$, and $\varphi_{\tau,\mathring{\mathbf{L}}_0}$ defines a perfect obstruction theory of $\mathscr{R}^{\mathring{\mathbf{L}}_0} \to \mathfrak{M}^{\mathring{\mathbf{L}}_0}$ compatible with φ_τ. Further pulling back to $\mathscr{R}^{\mathrm{cpt}}$, we obtain a perfect obstruction theory

$$\varphi_{\tau,\mathrm{cpt}} := \varphi_{\tau,\mathring{\mathbf{L}}_0}\big|_{\mathscr{R}^{\mathrm{cpt}}} \tag{2.51}$$

of $\mathscr{R}^{\mathrm{cpt}} \to \mathfrak{M}^{\mathrm{cpt}}$. Thus $\varphi_{\tau,\mathring{\mathbf{L}}_0}$ and $\varphi_{\tau,\mathrm{cpt}}$ are referred to as the *canonical perfect obstruction theories* of the corresponding morphisms.

2.6.3. *The canonical theory of decorated types.* Consider the decomposition

$$\mathscr{R}^{\mathrm{cpt}} = \bigsqcup_{\boldsymbol{\tau}} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}).$$

where the union runs through decorated types $\boldsymbol{\tau} = (\tau, \overline{\gamma}, \boldsymbol{\beta})$. In particular, we have

1. For any $x \in \mathbf{L}_0(G)$, $\mathbf{c}(x) = 0$ and $\overline{\gamma}_x$ is a 0-sector.
2. For any $x = \{h, \hat{h}\} \in \mathbf{E}(G)$ with $\mathbf{c}(x) = 0$, $\overline{\gamma}_x = (\overline{\gamma}_h, \overline{\gamma}_{\hat{h}})$ is a pair of $\mathfrak{P}$-sectors.

Fix a decorated type $\boldsymbol{\tau} = (\tau, \overline{\gamma}, \boldsymbol{\beta})$ as above. The restriction $\overline{\gamma}_{\mathbf{L}_0} := \overline{\gamma}|_{\mathbf{L}_0(G)}$ defines an open and closed substack $\mathfrak{M}^{\overline{\gamma}_{\mathbf{L}_0}} \subset \mathfrak{M}^{\mathrm{cpt}}$, over which gerbes corresponding to $\mathbf{L}_0(G)$ are given by zero sectors of $\overline{\gamma}_{\mathbf{L}_0}$. We thus have a factorization

$$\begin{tikzcd}
\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}) \arrow[rr] \arrow[dr] & & \mathfrak{M}^{\mathrm{cpt}} \\
& \mathfrak{M}^{\overline{\gamma}_{\mathbf{L}_0}} \arrow[ur] &
\end{tikzcd}$$

Pulling back (2.51), we obtain the *canonical perfect obstruction theory*

$$\varphi_{\boldsymbol{\tau}} := \varphi_{\tau,\mathrm{cpt}}|_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})}. \tag{2.52}$$

of $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}) \to \mathfrak{M}^{\overline{\gamma}_{\mathbf{L}_0}}$.

Let $\mathbf{S} \subset \mathbf{E}(G)$ be the set of edges with non-zero contact orders, and $\mathbf{L}_\infty(G) \subset \mathbf{L}(G)$ be the subset of legs with non-zero contact orders. Consider the evaluation stack

$$\mathfrak{M}^{\mathrm{ev}} := \mathfrak{M}^{\overline{\gamma}_{\mathbf{L}_0}} \times_{\mathfrak{M}} \prod_{x \in \mathbf{L}_\infty(G)} \mathfrak{M}^{\overline{\gamma}_x}(\mathcal{A}, \tau) \times_{\mathfrak{M}} \prod_{x \in \mathbf{S}} \mathfrak{M}^{\overline{\gamma}_x}(\mathcal{A}, \tau) \tag{2.53}$$

where all products are taken over $\mathfrak{M}$.[3] We obtain the tautological morphism

$$\mathbf{ev} \colon \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}) \to \mathfrak{M}^{\mathrm{ev}} \tag{2.54}$$

by taking the product of log evaluations along legs (2.44) and edges (2.45).

For the construction in §4, we introduce a perfect obstruction theory of $\mathbf{ev}$, compatible with the canonical one (2.52). Consider the universal family over $\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau})$

$$f_{\boldsymbol{\tau}} \colon C^{\circ}_{\boldsymbol{\tau}} \to \mathfrak{P}, \qquad \pi_{\boldsymbol{\tau}} \colon C^{\circ}_{\boldsymbol{\tau}} \to \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}).$$

Taking partial normalization along nodal gerbes $p_{\boldsymbol{\tau},x} \subset C^{\circ}_{\boldsymbol{\tau}}$ for all $x \in \mathbf{S}$, we obtain the strict morphism $\widetilde{C^{\circ}_{\boldsymbol{\tau}}} \to C^{\circ}_{\boldsymbol{\tau}}$. Composing the partial normalization with the universal arrows, we obtain the corresponding R-map and projection

$$\widetilde{f}_{\boldsymbol{\tau}} \colon \widetilde{C^{\circ}_{\boldsymbol{\tau}}} \to \mathfrak{P}, \qquad \widetilde{\pi}_{\boldsymbol{\tau}} \colon \widetilde{C^{\circ}_{\boldsymbol{\tau}}} \to \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau})$$

respectively. For each $x = \{h, \hat{h}\} \in \mathbf{S}(G)$, denote by $p_{\boldsymbol{\tau},h}, p_{\boldsymbol{\tau},\hat{h}} \subset \widetilde{C^{\circ}_{\boldsymbol{\tau}}}$ the pre-images of $p_{\boldsymbol{\tau},x}$ corresponding to the two half-edges. Set

$$\widetilde{\Sigma} = \sum_{x \in \mathbf{L}(G)} p_{\boldsymbol{\tau},x} + \sum_{x = \{h, \hat{h}\} \in \mathbf{S}} (p_{\boldsymbol{\tau},h} + p_{\boldsymbol{\tau},\hat{h}}),$$

and consider the complex

$$\mathbb{E}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})/\mathfrak{M}^{\mathrm{ev}}} := R\widetilde{\pi}_{\boldsymbol{\tau},*}\left(\widetilde{f}^*_{\boldsymbol{\tau}} T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\widetilde{\Sigma})\right). \tag{2.55}$$

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[3] Our notation $\mathfrak{M}^{\mathrm{ev}}$ is different from the same notation in [3] in that we do not include evaluations from nodal $\mathfrak{P}$-sectors.

By [3, Prop. 4.5], we obtain a morphism of distinguished triangles

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})/\mathfrak{M}^{\mathrm{ev}}} \arrow[r] \arrow[d, "\varphi_{\boldsymbol{\tau},\mathrm{ev}}"'] & \mathbb{T}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})/\mathfrak{M}^{\overline{\gamma}\mathbf{L}_0}} \arrow[r] \arrow[d, "\varphi_{\boldsymbol{\tau}}"'] & T_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}^{\overline{\gamma}\mathbf{L}_0}}|_{\mathscr{R}} \arrow[r, "{[1]}"] \arrow[d, equal] & {} \\
\mathbb{E}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})/\mathfrak{M}^{\mathrm{ev}}} \arrow[r] & \mathbb{E}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})/\mathfrak{M}^{\overline{\gamma}\mathbf{L}_0}} \arrow[r] & T_{\mathfrak{M}^{\mathrm{ev}}/\mathfrak{M}^{\overline{\gamma}\mathbf{L}_0}}|_{\mathscr{R}} \arrow[r, "{[1]}"] & {}
\end{tikzcd} \tag{2.56}$$

where $\varphi_{\boldsymbol{\tau}}$ is the canonical perfect obstruction theory (2.52), and $\varphi_{\boldsymbol{\tau},\mathrm{ev}}$ is a perfect obstruction theory relative to $\mathfrak{M}^{\mathrm{ev}}$. In particular, the two perfect obstruction theories $\varphi_{\boldsymbol{\tau}}$ and $\varphi_{\boldsymbol{\tau},\mathrm{ev}}$ are compatible in the sense of [8, §7]. Thus, we call $\varphi_{\boldsymbol{\tau},\mathrm{ev}}$ the *canonical perfect obstruction theory* relative to $\mathfrak{M}^{\mathrm{ev}}$.

Assume that τ is realizable, then $\mathfrak{M}$ is of pure-dimension by Proposition B.28. By Lemma 2.9 and §2.5.4, we obtain the pure-dimensionality of $\mathfrak{M}^{\mathrm{ev}}$. In this case, the canonical perfect obstruction theories $\varphi_{\boldsymbol{\tau}}$ and $\varphi_{\boldsymbol{\tau},\mathrm{ev}}$ define the same virtual cycle $[\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})]^{\mathrm{vir}}$, called the *canonical virtual cycle* of $\mathscr{R}(\mathfrak{P},\boldsymbol{\tau})$.

3. MODULAR PRINCIPALIZATION VIA UNIFORM MAXIMAL DEGENERACIES

To construct the reduced perfect obstruction theory, it is essential to understand how the Kiem–Li cosection extends across the logarithmic boundary of the moduli of punctured R-maps. In particular, we need to understand the pole order of cosections along each part of the boundary. Unfortunately, the logarithmic boundary is virtually a Weil divisor, leading to difficulties in understanding the boundary behavior of cosections. The key to resolving this issue is the principalization developed in [20, 19] parameterizing log maps with *uniform maximal degeneracies*. In this section, we further develop this theory in the case of punctured R-maps, needed in log GLSM calculations.

3.1. **Punctured maps with uniform maximal degeneracies.**

3.1.1. *The uniform maximal degeneracy.* A tropical map

$$(\mathbf{G}, \ell, f^{\mathrm{trop}}\colon \Gamma(G,\ell) \to \mathbb{R}_{\geq 0}), \ \text{ with } \mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}) \tag{3.1}$$

over σ (§B.17) is said to have the *uniform maximal degeneracy*, if the set of degeneracies (see §B.2.6) $\{e_V\}_{V\in\mathbf{V}(G)}$ has a maximum, denoted by $e_{\max}(f^{\mathrm{trop}})$. When there is no danger of confusion, we may write $e_{\max}$ for the maximal degeneracy and omit f^{trop}. Such a tropical map is also referred to as a $\curlywedge$-*tropical map.*

The $\curlywedge$-*type* of a $\curlywedge$-tropical map f^{trop} over σ is

$$(\tau, \mathbf{V}_{\max}(G)) \tag{3.2}$$

consisting of a type $\tau = (\mathbf{G}, \boldsymbol{\sigma}, \mathbf{c})$ as in (B.19), and a non-empty set

$$\mathbf{V}_{\max}(G) := \{V \in \mathbf{V}(G) \mid e_V = e_{\max}\}.$$

A punctured map $\mathfrak{f}\colon C^{\circ} \to \mathcal{A}$ over a geometric log point S is said to have the *uniform maximal degeneracy* if its corresponding tropical map has the uniform maximal degeneracy. We may call such $\mathfrak{f}$ a $\curlywedge$-*punctured map* for simplicity. The *tropical* $\curlywedge$-*type* of $\mathfrak{f}$ is the $\curlywedge$-type of the associated $\curlywedge$-tropical map.

3.1.2. *Basic* $\curlywedge$-*tropical maps.*

Definition 3.1. A $\curlywedge$-tropical map (with not necessarily connected domain) is $\curlywedge$-*basic* if it is universal among all $\curlywedge$-tropical maps of the same $\curlywedge$-type (c.f. Definition B.10).

Construction 3.2. For a $\curlywedge$-tropical map (3.1) over $\sigma \in \mathbf{Cones}$ of type (3.2), we construct its associated $\curlywedge$-basic tropical map as follows.

First, let $f^{\mathrm{trop}}_{bas}\colon \Gamma(G,\ell_{bas}) \to \mathbb{R}_{\geq 0}$ over σ_{bas} be the basic map associated to f^{trop} among all tropical maps of type $(\mathbf{G},\boldsymbol{\sigma},\mathbf{c})$, see §B.3.1. Following (B.23), we define the $\curlywedge$-basic cone $\sigma_{\curlywedge} \subset \sigma_{bas}$ to be the sub-cone consisting of vectors $((p_V)_V,(e_E)_E) \in \sigma_{bas}$ satisfying

(1) $p_V = p_{V'}$ for any $V, V' \in \mathbf{V}_{\max}(G)$,
(2) $p_V \geq p_{V'}$ for any $V \in \mathbf{V}_{\max}(G)$ and $V' \notin \mathbf{V}_{\max}(G)$.

Denote by $f^{\mathrm{trop}}_{\curlywedge}\colon \Gamma(G,\ell_{\curlywedge}) \to \mathbb{R}_{\geq 0}$ the pull-back of f^{trop}_{bas} to $\sigma_{\curlywedge} \subset \sigma_{bas}$. In particular, we have a canonical surjective morphism of monoids

$$\sigma^{\vee}_{bas,\mathbb{Z}} \to \sigma^{\vee}_{\curlywedge,\mathbb{Z}}. \tag{3.3}$$

In case there's no punctured legs, $\sigma^{\vee}_{\curlywedge,\mathbb{Z}}$ is the *minimal monoid* in [20, Definition 3.11]

Proposition 3.3. *$f^{\mathrm{trop}}_{\curlywedge}$ is the $\curlywedge$-basic tropical map associated to f^{trop}.*

Proof. By the universality of f^{trop}_{bas} (see §B.3.1), there is a natural morphism $\sigma \to \sigma_{bas}$ such that h is the pull-back of f^{trop}_{bas}. To see the universality of $f^{\mathrm{trop}}_{\curlywedge}$, it suffices to notice that the morphism $\sigma \to \sigma_{bas}$ factors through $\sigma_{\curlywedge} \subset \sigma_{bas}$. This finishes the proof. □

3.1.3. *Rigid $\curlywedge$-basic tropical maps.*

For a $\curlywedge$-tropical map over $\sigma \in \mathbf{Cones}$ as in (3.1), observe that $e_{\max} \in \sigma^{\vee}_{\mathbb{Z}}$ is a morphism of cones

$$e_{\max}\colon \sigma \to \mathbb{R}_{\geq 0}. \tag{3.4}$$

Note that when $e_{\max} = 0$, the image of $e_{\max}$ is the zero cone $0 \in \mathbb{R}_{\geq 0}$.

Definition 3.4. A $\curlywedge$-basic tropical map with connected domain is said to be *rigid* if the morphism of cones (3.4) is injective.

For a rigid $\curlywedge$-basic tropical map $(G,\mathbf{g},\mathbf{deg},\mathbf{m},\ell,f^{\mathrm{trop}})$ over σ, one of the following holds:

(1) $\sigma = \{0\}$, in which case the rigid $\curlywedge$-tropical map is *non-degenerate*;
(2) $\sigma \cong \mathbb{R}_{\geq 0}$, in which case the rigid $\curlywedge$-tropical map is *degenerate*.

In the non-degenerate case $\sigma = \{0\}$, the $\curlywedge$-basicness implies that

$$|\mathbf{V}(G)| = 1, \qquad \mathbf{V}_{\max}(G) = \mathbf{V}(G), \qquad \mathbf{E}(G) = \emptyset. \tag{3.5}$$

In the degenerate case $\sigma \cong \mathbb{R}_{\geq 0}$, we have a description similar to [2, Prop. 5.1]:

Proposition 3.5. *Suppose that $(G,\mathbf{g},\mathbf{deg},\mathbf{m},\ell,f^{\mathrm{trop}})$ is a degenerate rigid $\curlywedge$-tropical map over $\sigma \cong \mathbb{R}_{\geq 0}$. Then we have*

(1) f^{trop} contracts no edges.
(2) $\mathbf{V}_{\max}(G)$ is precisely the collection of degenerate vertices in $\mathbf{V}(G)$.

In this case viewing $e_{\max} \in \sigma^{\vee}_{\mathbb{Z}} \cong \mathbb{N}$ as a positive integer, we have

$$e_{\max} = \mathrm{lcm}_{x \in \mathbf{E}(G)}(\mathbf{c}(x)). \tag{3.6}$$

Proof. First note that if an edge $x \in \mathbf{E}(G)$ is contracted, its edge length $\ell(x)$ can be arbitrary. Thus by $\curlywedge$-basicness, the morphism $e_{\max}\colon \sigma \to \mathbb{R}_{\geq 0}$ fails to be injective.

Now suppose a vertex $V \in \mathbf{V}(G)$ with degeneracy e_V such that $0 \prec e_V \prec e_{\max}$. We may produce a family f^{trop}_t of tropical maps for t a real number close to 0, with $f^{\mathrm{trop}}_0 = f^{\mathrm{trop}}$, $f^{\mathrm{trop}}_t|_{\sigma_{V'}} = f^{\mathrm{trop}}|_{\sigma_{V'}}$ for $V' \neq V$, and $f^{\mathrm{trop}}_t|_{\sigma_V} = f^{\mathrm{trop}}|_{\sigma_V} + t$. By doing so, we will need to modify the lengths of edges attached to V to fix $e_{\max}$. This implies that $\dim \sigma > 1$, hence h is not rigid. This finishes the proof. □

3.1.4. *Basicness of $\curlywedge$-punctured maps to $\mathcal{A}$.*

Definition 3.6. A $\curlywedge$-punctured map (with not necessarily connected domain) over an fs log scheme is *$\curlywedge$-basic* if for every geometric fiber, the associated tropical map is $\curlywedge$-basic.

Proposition 3.7. *For a family of punctured maps $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over an fs log scheme S, the locus $S_{bas} \subset S$ consisting of $\curlywedge$-basic fibers is open in S.*

Proof. The case of $\curlywedge$-basic log maps is established in [20, Prop. 3.13]. Note that the constructions of both basic monoids and $\curlywedge$-basic monoid involve only degeneracies of vertices, edge lengths and contact orders, regardless of information about the markings. The case of $\curlywedge$-basic punctured maps thus follows by an identical proof. $\square$

The following is an analogue of Proposition B.15:

Proposition 3.8. *Any pre-stable $\curlywedge$-punctured maps to $\mathcal{A}$ is a pull-back from a $\curlywedge$-basic pre-stable punctured map to $\mathcal{A}$ with the same underlying pre-stable map. Both the $\curlywedge$-basic map and the morphism are unique up to a unique isomorphism.*

Proof. The universality of $\curlywedge$-basic of log maps is established in [20, Prop. 3.13]. Below, we adapt the proof to also apply to more general $\curlywedge$-punctured maps.

Given a pre-stable $\curlywedge$-punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S, we first take the corresponding basic object $\mathfrak{f}_{bas}\colon C^\circ_{bas} \to \mathcal{A}$ over S_{bas}, constructed via Proposition B.15. Using the universal property of $\curlywedge$-basicness in Definition 3.1, the method in the proof of [20, Prop. 3.13] produces a morphism of log schemes $S_\curlywedge \to S_{bas}$ with isomorphic underlying $\underline{S}_\curlywedge \cong \underline{S}_{bas} \cong \underline{S}$ such that $\mathfrak{f}_{bas}$ pulls back to a $\curlywedge$-punctured map $f_\curlywedge\colon C^\circ_\curlywedge \to \mathcal{A}$ over $\underline{S}_\curlywedge$, which satisfies $\curlywedge$-basicness at a point $s \in S$. Note that $\curlywedge$-basicness is an open condition thanks to Proposition 3.7. Thus the family $f_\curlywedge$ over $S_\curlywedge$ is at least $\curlywedge$-basic in a neighborhood of s. As the statement is local on S, shrinking S we may assume $f_\curlywedge$ is $\curlywedge$-basic. Finally, the proof of [20, Prop. 3.13] also shows that the universal property in Definition 3.1 implies that the canonical map $S \to S_{bas}$ factors through $S_\curlywedge \to S_{bas}$ uniquely. Hence $\mathfrak{f}$ is the pull-back of $\mathfrak{f}_\curlywedge$ as desired. $\square$

3.2. **Stacks of punctured maps to $\mathcal{A}$ with uniform maximal degeneracies.**

3.2.1. *Contractions of $\curlywedge$-types.* A *contraction* of $\curlywedge$-types

$$\phi\colon \tau'_\curlywedge = (\tau', \mathbf{V}_{\max}(G')) \to \tau_\curlywedge = (\tau, \mathbf{V}_{\max}(G)) \tag{3.7}$$

is a contraction of types $\phi\colon \tau' \to \tau$ as in §B.3.2, further satisfying

$$\phi(\mathbf{V}_{\max}(G')) \subset \mathbf{V}_{\max}(G). \tag{3.8}$$

Following Notation B.18, we may use $\tau'_\curlywedge \vdash \tau_\curlywedge$ to denote a contraction $\tau'_\curlywedge \to \tau_\curlywedge$.

Let $\phi : \tau'_\curlywedge \to \tau_\curlywedge$ be a contraction. Denote by $\sigma'_\curlywedge$ and $\sigma_\curlywedge$ the $\curlywedge$-basic cones of $\tau'_\curlywedge$ and $\tau_\curlywedge$ respectively. By §3.1.2 they are subcones $\sigma'_\curlywedge \subset \sigma'$ and $\sigma_\curlywedge \subset \sigma$ of the corresponding basic cones of τ' and τ. Observe that (B.26) restricts to a face morphism

$$\chi^\vee_\phi\colon \sigma_\curlywedge \to \sigma'_\curlywedge. \tag{3.9}$$

Indeed (1), (2) in Construction 3.2 are preserved by (3.9) thanks to (3.8). Thus $\sigma_\curlywedge$ is identified with the intersection $\sigma'_\curlywedge \cap \sigma$ in σ'. Taking dual, we obtain

$$\chi_\phi\colon (\sigma'_{\curlywedge,\mathbb{Z}})^\vee \to \sigma^\vee_{\curlywedge,\mathbb{Z}} \tag{3.10}$$

satisfying the same conditions §B.3.2 ($1^\vee$), ($2^\vee$).

For a contraction (3.7), we define the *$\curlywedge$-basic monoid ideal* to be

$$\overline{\mathcal{K}}^\curlywedge_\phi := \chi^{-1}_\phi(\sigma^\vee_{\curlywedge,\mathbb{Z}} \setminus \{0\}). \tag{3.11}$$

It is compatible with the basic monoid ideal (B.28) in the following sense.

Lemma 3.9. *Suppose $\tau_\curlywedge$ is realizable. Then, we have*

(1) *$\mathring{\sigma}_{\curlywedge} \subset \mathring{\sigma}$, where $\mathring{\sigma}_{\curlywedge}$ and $\mathring{\sigma}$ are the interiors of $\sigma_{\curlywedge}$ and σ respectively.*
(2) *The following diagram is commutative*

$$\begin{tikzcd}
\sigma^{\vee}_{\mathbb{Z}} \setminus \{0\} \arrow[r, "\subset"] \arrow[d] & \sigma^{\vee}_{\mathbb{Z}} \arrow[d] \\
\sigma^{\vee}_{\curlywedge,\mathbb{Z}} \setminus \{0\} \arrow[r, "\subset"] & \sigma^{\vee}_{\curlywedge,\mathbb{Z}}
\end{tikzcd}$$

In particular, the image of the basic monoid ideal $\overline{\mathcal{K}}_{\phi} = \chi_{\phi}^{-1}(\sigma^{\vee}_{\mathbb{Z}} \setminus \{0\})$ via $\sigma^{\vee}_{\mathbb{Z}} \to \sigma^{\vee}_{\curlywedge,\mathbb{Z}}$ is contained in the $\curlywedge$-basic monoid ideal $\overline{\mathcal{K}}^{\curlywedge}_{\phi}$.

Proof. Suppose that $\mathring{\sigma}_{\curlywedge}$ is contained in a proper face $\sigma' \subset \sigma$. Then as in [3, §2.2.2], the generization $\sigma_{\mathbb{Z}} \to \sigma'_{\mathbb{Z}}$ defines a contraction of types $\tau \to \tau'$. Hence τ' is the associated type of τ_{σ}, leading to a contradiction.

Note that $\sigma^{\vee}_{\mathbb{Z}} \setminus \{0\}$ are precisely elements in $\sigma^{\vee}_{\mathbb{Z}}$, which are strictly positive along $\mathring{\sigma}$. Thus the image of $\sigma^{\vee}_{\mathbb{Z}} \setminus \{0\}$ along the right vertical arrow are collection of linear functions strictly positive along $\mathring{\sigma}_{\curlywedge}$ thanks to (1). In particular, the image of $\sigma^{\vee}_{\mathbb{Z}} \setminus \{0\}$ is in $\sigma^{\vee}_{\curlywedge,\mathbb{Z}} \setminus \{0\}$, proving (2). ☐

3.2.2. *Marking by $\curlywedge$-types.*

Definition 3.10. A *$\tau_{\curlywedge}$-marking* of a $\curlywedge$-basic tropical map f^{trop} by a $\curlywedge$-type $\tau_{\curlywedge}$ is a contraction of $\curlywedge$-types $\phi \colon \tau_{f^{\mathrm{trop}}} \to \tau_{\curlywedge}$, where $\tau_{f^{\mathrm{trop}}}$ is the $\curlywedge$-type of f^{trop}.

Definition 3.11. Let $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$ be a $\curlywedge$-type, and let $\mathfrak{f} \colon C^{\circ} \to \mathcal{A}$ be a $\curlywedge$-basic punctured map over S. For any geometric point $s \in S$, let $\tau_{\curlywedge,s} = (\tau_s, \mathbf{V}_{\max}(G_s))$ be the tropical $\curlywedge$-type of the fiber $\mathfrak{f}|_s$. We say that $\mathfrak{f}$ is *marked by $\tau_{\curlywedge}$* if

(1) Its associated basic map is marked by τ §B.3.3.
(2) The contraction $\tau_s \to \tau$ given by the associated basic maps induces a contraction of $\curlywedge$-types $\phi_{\tau_{\curlywedge,s},\tau_{\curlywedge}} : \tau_{\curlywedge,s} \to \tau_{\curlywedge}$, as in §3.2.1. In particular, the $\curlywedge$-basic tropical map of the fiber $\mathfrak{f}|_s$ is marked by $\tau_{\curlywedge}$.
(3) For any geometric point $s \in S$, the log-ideal $\mathcal{K}^{\curlywedge}_{\phi_{\tau_{\curlywedge,s},\tau_{\curlywedge}}} \subset \mathcal{M}_{S,s}$ defined as the pre-image of the $\curlywedge$-basic monoid ideal $\overline{\mathcal{K}}^{\curlywedge}_{\phi_{\tau_{\curlywedge,s},\tau_{\curlywedge}}} \subset \overline{\mathcal{M}}_{S,s}$ maps to 0 under the structural morphism $\mathcal{M}_{S,s} \to \mathcal{O}_{S,s}$.

Similar to [3, Definition 3.20], the collection of stalks $\mathcal{K}^{\curlywedge}_{\phi_{\tau_{\curlywedge,s},\tau_{\curlywedge}}} \subset \mathcal{M}_{S,s}$ in Definition 3.11 (3) form a coherent ideal $\mathcal{K}^{\curlywedge}_{S} \subset \mathcal{M}_S$, called the *$\tau_{\curlywedge}$-marking ideal.* Recall that the base S is also equipped with an idealized structure $\mathcal{K}_S$ from the puncturing ideal and the pull-back of the canonical idealized structure in (B.31) via the basic family induced by the τ-marking.

Proposition 3.12. *Suppose the $\curlywedge$-type $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$ is realizable, and hence τ is also realizable. Then there is a natural inclusion of sheaves of log-ideals $\mathcal{K}_S \hookrightarrow \mathcal{K}^{\curlywedge}_S$.*

Proof. It suffices to check the inclusion on the stalk level of the corresponding characteristic sheaves $\overline{\mathcal{K}}_{S,s} \hookrightarrow \overline{\mathcal{K}}^{\curlywedge}_{S,s}$ for any geometric point $s \in S$. Since τ is realizable, Lemma B.22 implies that $\overline{\mathcal{K}}_S$ is fiberwise given by the basic monoid ideal. By Lemma 3.9 (2), the image of the basic monoid ideal is contained in the $\curlywedge$-basic monoid ideal. This completes the proof. ☐

3.2.3. *Stacks of $\curlywedge$-punctured maps.* Fix a type $\tau = (\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}), \boldsymbol{\sigma}, \mathbf{c})$, with possibly disconnected G. Consider the categories fibered over fs log schemes

$$\mathfrak{U}(\mathcal{A}, \tau), \qquad \mathfrak{U}'(\mathcal{A}, \tau), \qquad \mathfrak{M}(\mathcal{A}, \tau), \qquad \mathfrak{M}'(\mathcal{A}, \tau),$$

where the left two categories parameterize $\curlywedge$-punctured maps whose corresponding basic families are marked and weakly marked by τ, and the right two are log stacks of basic punctured maps marked and weakly marked by τ respectively as in §B.3.4.

Proposition 3.13. *The tautological morphisms*

$$\mathfrak{U}(\mathcal{A},\tau) \to \mathfrak{M}(\mathcal{A},\tau), \qquad \mathfrak{U}'(\mathcal{A},\tau) \to \mathfrak{M}'(\mathcal{A},\tau) \tag{3.12}$$

by taking the corresponding basic families, are proper, log étale and representable by log algebraic spaces. In particular, both $\mathfrak{U}(\mathcal{A},\tau)$ and $\mathfrak{U}'(\mathcal{A},\tau)$ are log algebraic stacks of locally finite type with their $\curlywedge$-basic log structures.

Remark 3.14. Denote by $\mathcal{K}_{\mathfrak{M}(\mathcal{A},\tau)}$ and $\mathcal{K}_{\mathfrak{M}'(\mathcal{A},\tau)}$ the canonical idealized structures on $\mathfrak{M}(\mathcal{A},\tau)$ and $\mathfrak{M}'(\mathcal{A},\tau)$ respectively, see (B.31). Pulling back along (3.12), we obtain the corresponding *canonical idealized structures* $\mathcal{K}_{\mathfrak{U}(\mathcal{A},\tau)}$ and $\mathcal{K}_{\mathfrak{U}'(\mathcal{A},\tau)}$ on $\mathfrak{U}(\mathcal{A},\tau)$ and $\mathfrak{U}'(\mathcal{A},\tau)$ respectively. In particular, the morphisms in (3.12) become ideally strict morphisms of idealized log stacks

$$\begin{aligned}(\mathfrak{U}(\mathcal{A},\tau),\mathcal{K}_{\mathfrak{U}(\mathcal{A},\tau)}) &\to (\mathfrak{M}(\mathcal{A},\tau),\mathcal{K}_{\mathfrak{M}(\mathcal{A},\tau)}),\\ (\mathfrak{U}'(\mathcal{A},\tau),\mathcal{K}_{\mathfrak{U}'(\mathcal{A},\tau)}) &\to (\mathfrak{M}'(\mathcal{A},\tau),\mathcal{K}_{\mathfrak{M}'(\mathcal{A},\tau)}).\end{aligned} \tag{3.13}$$

By [31, IV. Variant 3.1.22] and Proposition 3.13, these two morphisms are idealized log étale.

Proof. The statement for log maps is established in [20, Theorem 3.17]. We explain below how the identical proof also applies to general punctured maps. For what follows, we use the following shorthand notations for simplicity:

$$\mathfrak{U} = \mathfrak{U}(\mathcal{A},\tau), \qquad \mathfrak{U}' = \mathfrak{U}'(\mathcal{A},\tau), \qquad \mathfrak{M} = \mathfrak{M}(\mathcal{A},\tau), \qquad \mathfrak{M}' = \mathfrak{M}'(\mathcal{A},\tau).$$

Step 1: Representability and log étaleness. Consider Olsson's log stack $\mathrm{Log}_{\mathfrak{M}}$, which associates to any $\underline{\mathfrak{M}}$-scheme $\underline{T}$ the category of log maps $T \to \mathfrak{M}$ of fs log stacks with underlying $\underline{T} \to \underline{\mathfrak{M}}$.[4] As argued in [20, §3.3.2], since having uniform maximal degeneracy is a configuration of degeneracies in basic monoids, the tautological morphism $\mathfrak{U} \to \mathfrak{M}$ factors through the natural projection $\mathrm{Log}_{\mathfrak{M}} \to \mathfrak{M}$. Proposition 3.7 further implies that the morphism $\mathfrak{U} \to \mathrm{Log}_{\mathfrak{M}}$ is an open embedding. Since $\mathrm{Log}_{\mathfrak{M}}$ is algebraic and locally of finite type by [32, Thm. 1.1], the same properties hold for $\mathfrak{U}$. Since (3.3) is surjective, the same proof as in [20, Prop. 3.16] implies that the morphism $\mathfrak{U} \to \mathfrak{M}$ is representable. The log étaleness of $\mathfrak{U} \to \mathfrak{M}$ then follows from [32, Thm. 4.6 (ii), (iii)]. The same argument applies to the case of weak markings $\mathfrak{U}' \to \mathfrak{M}'$.

Step 2: Finite type. Next, we argue that the morphism $\mathfrak{U} \to \mathfrak{M}$ is of finite type. Consider any strict morphism $T \to \mathfrak{M}$ and $U := T \times_{\mathfrak{M}} \mathfrak{U}$. It suffices to show that the projection $U \to T$ is of finite type. Since the question is local on T, we may further assume that T is of finite type. Consequently, punctured maps parameterized by T have finitely many tropical types. Note that for each tropical type given by a fiber of T, the set of possible $\curlywedge$-types, determined by the choices of subsets with maximal degenerate vertices, is finite. Then the identical argument in [20, §3.3.2] constructs a finite-type cover of U. Again, the same argument applies to the case of weak markings $\mathfrak{U}' \to \mathfrak{M}'$.

Step 3: Properness. It remains to show that $\mathfrak{U} \to \mathfrak{M}$ and $\mathfrak{U}' \to \mathfrak{M}'$ satisfy the valuative criterion. As the proof in both cases is identical to [20, §3.3.3], we will only sketch the proof in the case of markings by τ, and refer to [20, §3.3.3] for details.

To set up the problem, let R be a discrete valuation ring with maximal ideal $\mathfrak{m} \subset R$ and and quotient field K. Consider a commutative diagram of solid arrows of the underlying

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[4] As our base log structures are fs, the stack $\mathrm{Log}_{\mathfrak{M}}$ denotes the open substack parameterizing fs log structures over $\mathfrak{M}$.

stacks

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r, "{[\mathfrak{f}^{\Yup}_{\eta}]}"] \arrow[d] & \underline{\mathfrak{U}} \arrow[d] \\
\operatorname{Spec} R \arrow[r, "{[\mathfrak{f}]}"'] \arrow[ur, dashed, "{[\mathfrak{f}^{\Yup}]}"] & \underline{\mathfrak{M}}
\end{tikzcd}$$

It suffices to show that possibly after replacing R by a finite extension of discrete valuation rings, and replacing K by the corresponding finite extension of quotient fields, there exists a unique dashed arrow making the above diagram commutative. In the diagram, $\mathfrak{f}$ is the basic punctured map over $S = (\operatorname{Spec} R, \mathcal{M}_S)$ given by the bottom arrow, and $\mathfrak{f}^{\Yup}_{\eta}$ is the $\Yup$-basic punctured map over $\eta_{\Yup} = (\operatorname{Spec} K, \mathcal{M}_{\eta_{\Yup}})$ given by the top arrow. Denote by $s, \eta \in S$ the closed and generic points with pull-back log structures from S respectively. Denote by $\mathfrak{f}_s, \mathfrak{f}_\eta$ the fibers of $\mathfrak{f}$ over s, η respectively. Thus, there is a morphism $\eta_{\Yup} \to \eta$ such that $\mathfrak{f}_\eta$ pulls back to $\mathfrak{f}^{\Yup}_{\eta}$. To construct the dashed arrow, it amounts to construct a $\Yup$-basic punctured map $\mathfrak{f}^{\Yup}$ extending $\mathfrak{f}^{\Yup}_{\eta}$ across the closed point of $\operatorname{Spec} R$. Denote by $\mathfrak{f}^{\Yup}_{s}$ the closed fiber of $\mathfrak{f}^{\Yup}$.

Since having uniform maximal degeneracy does not modify the underlying pre-stable map, we may define the underlying morphism via $\underline{\mathfrak{f}}^{\Yup} = \underline{\mathfrak{f}}$ over $\operatorname{Spec} R$. Denote by τ_s the tropical type of $\mathfrak{f}_s$. By [20, §3.3.3, Step 2], the generic fiber $\mathfrak{f}^{\Yup}_{\eta}$ and the underlying morphism $\underline{\mathfrak{f}}^{\Yup}$ together determine a unique tropical $\Yup$-type $\tau_{\Yup,s}$ of the central fiber $\mathfrak{f}^{\Yup}_{s}$ provided its existence. Indeed, given τ_s it remains to specify the collection of maximally degenerate vertices $\mathbf{V}_{\max}(G)$ to obtain $\tau_{\Yup,s}$. The proof in [20, §3.3.3, Step 2] does not rely on information from punctures, hence can be applied here identically.

To construct the limit fiber $\mathfrak{f}^{\Yup}_{s}$, it suffices to modify $\mathfrak{f}_s$ to satisfy §3.1.2 (1), (2) tropically. Condition §3.1.2 (1) is achieved in [20, §3.3.3, Step 3], and §3.1.2 (2) is achieved in [20, §3.3.3, Step 4]. Indeed [20, §3.3.3, Step 3, Step 4] shows that both conditions are achieved by a sequence of log blow-ups [31, III, 2.6] of S determined by $\mathbf{V}_{\max}(G)$ of the $\Yup$-type $\tau_{\Yup,s}$. As these log blow-ups do not using information from the punctures, they can be applied in the punctured case identically, leading to the desired limit fiber $\mathfrak{f}^{\Yup}_{s}$.

Finally, applying [20, §3.3.3, Step 5], we see that the construction of $\mathfrak{f}^{\Yup}_{s}$ is independent of the choice of sequence of log blow-ups thanks to the universal properties of log blow-ups (see [31, III, Def. 2.6.2]) and $\Yup$-basicness (see Definition 3.1).

This finishes the proof. □

Proposition 3.15. *There is a Cartesian diagram of log stacks*

$$\begin{tikzcd}
\mathfrak{U}(\mathcal{A}, \tau) \arrow[r] \arrow[d] & \mathfrak{U}'(\mathcal{A}, \tau) \arrow[d] \\
\mathfrak{M}(\mathcal{A}, \tau) \arrow[r] & \mathfrak{M}'(\mathcal{A}, \tau)
\end{tikzcd}$$

with vertical arrows given by (3.12)*, and the bottom arrow given by* (B.35)*. In particular, both horizontal arrows are a strict closed embedding defined by a nilpotent ideal.*

Proof. It follows from the definitions that the square is Cartesian of log stacks. Since the two horizontal arrows are strict, the diagram of the underlying stacks is also Cartesian. By Proposition B.27 the bottom arrow is a strict closed embedding defined by a nilpotent ideal. Hence the same property holds for the top arrow. □

We now consider a $\Yup$-type $\tau_{\Yup} = (\tau, \mathbf{V}_{\max}(G))$. Let $\mathfrak{U}(\mathcal{A}, \tau_{\Yup})$ be the category of $\Yup$-basic punctured maps to $\mathcal{A}$ marked by $\tau_{\Yup}$. By Definition 3.11 there is a tautological morphism

$$\mathfrak{U}(\mathcal{A}, \tau_{\Yup}) \to \mathfrak{U}(\mathcal{A}, \tau). \tag{3.14}$$

We are interested in the realizable case.

Proposition 3.16. *Suppose that* $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$ *hence* τ *are realizable. Then*

(1) The morphism (3.14) *is a strict closed embedding. In particular,* $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ *is a log algebraic stack locally of finite type.*

(2) $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ *admits a canonical idealized structure given by the* $\tau_{\curlywedge}$*-marking ideal* $\mathcal{K}_{\mathfrak{U}(\mathcal{A},\tau_{\curlywedge})}$*. If we further equip* $\mathfrak{U}(\mathcal{A}, \tau)$ *with its canonical idealized structure* $\mathcal{K}_{\mathfrak{U}(\mathcal{A},\tau)}$ *as in Remark 3.14, then* (3.14) *is an idealized log étale morphism of idealized log stacks*

$$(\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}), \mathcal{K}_{\mathfrak{U}(\mathcal{A},\tau_{\curlywedge})}) \to (\mathfrak{U}(\mathcal{A}, \tau), \mathcal{K}_{\mathfrak{U}(\mathcal{A},\tau)}). \tag{3.15}$$

Proof. We divide the proof into several steps.

Step 1. Set-up for a local comparison. For any strict morphism $S \to \mathfrak{U}(\mathcal{A}, \tau)$, we will establish the statements by studying the projection $S' = S \times_{\mathfrak{U}(\mathcal{A},\tau)} \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \to S$. As both statements can be checked locally on $\mathfrak{U}(\mathcal{A}, \tau)$, hence locally on S, we may replacing S by any étale chart as needed.

Let $\mathfrak{f}_{\curlywedge} \colon C^{\circ} \to \mathcal{A}$ be the $\curlywedge$-basic punctured map over S, with the associated basic map $\mathfrak{f} \colon C^{\circ}_b \to \mathcal{A}$ over S_b. For any log point $t \in S$, denote by $t_b \in S_b$ its image, where both t and t_b are equipped with the pull-back log structures from S and S_b respectively.

Consider the $\curlywedge$-basic tropical map $\Sigma(\mathfrak{f}_{\curlywedge}|_t) \colon \Gamma(G_t, \ell_t) \to \mathbb{R}_{\geq 0}$ over $\sigma_t = \overline{\mathcal{M}}_t^{\vee}$ by taking the tropicalization of the fiber $\mathfrak{f}_{\curlywedge}|_t$ over t. Let $\tau_{\curlywedge,t} = (\tau_t, \mathbf{V}_{\max}(G_t))$ be the $\curlywedge$-type of $\Sigma(\mathfrak{f}_{\curlywedge}|_t)$. Similarly, the associated basic fiber $\mathfrak{f}|_{t_b}$ over t_b has the basic tropical map by taking tropicalization $\Sigma(\mathfrak{f}|_{t_b}) \colon \Gamma(G_t, \ell_{b,t_b}) \to \mathbb{R}_{\geq 0}$ over $\sigma_{t_b} = \overline{\mathcal{M}}_{t_b}^{\vee}$ with type τ_t. Since $\mathfrak{f}$ is marked by τ, there is a contraction $\phi_t \colon \tau_t \to \tau$ of types for any geometric point $t \in S$. If further $\phi_t(\mathbf{V}_{\max}(G_t)) \subset \mathbf{V}_{\max}(G)$, then $\mathfrak{f}_{\curlywedge}|_t$ is marked by $\tau_{\curlywedge}$ by Definition 3.11.

Step 2. Local construction of the $\tau_{\curlywedge}$-marking loci. Suppose there exists a geometric point $s \in S$ such that $\mathfrak{f}_{\curlywedge}|_s$ is marked by $\tau_{\curlywedge}$. Otherwise, we have $S' = \emptyset$, hence nothing to prove. Shrinking S if necessary, we may assume there are charts of log structures

$$\overline{\mathcal{M}}_{S,s} \to \Gamma(S, \mathcal{M}_S), \qquad \overline{\mathcal{M}}_{S_b,s_b} \to \Gamma(S, \mathcal{M}_{S_b}) \tag{3.16}$$

Let $\overline{\mathcal{K}}_s^{\curlywedge} \subset \overline{\mathcal{M}}_{S,s}$ be the $\curlywedge$-basic monoid ideal as in §3.2.1, and denote by $\mathcal{K}_S^{\curlywedge} \subset \mathcal{M}_S$ the log-ideal generated by the image of $\overline{\mathcal{K}}_s^{\curlywedge}$ using the chart

$$\overline{\mathcal{M}}_{S,s} \to \Gamma(S, \mathcal{M}_S) \to \mathcal{M}_S.$$

Let $S'' \subset S$ be the strict closed subscheme cut out by $\alpha_S(\mathcal{K}_S^{\curlywedge}) = 0$. We will show that $S' = S''$. In particular, as $S' \subset S$ is cut out by $\alpha_S(\mathcal{K}_S^{\curlywedge})$, the idealized log étaleness in (2) of the statement follows from [31, IV. Variant 3.1.21].

Step 3: Contractions of $\curlywedge$-types. For any geometric point $t \in S$, the composition via the chart in (3.16)

$$\overline{\mathcal{M}}_{S_b,s_b} \to \mathcal{M}_{S_b} \to \overline{\mathcal{M}}_{S_b,t_b}$$

is a generization morphism of characteristic monoids. Taking dual of the generization, we obtain a face morphism of basic cones $\sigma_t \to \sigma_s$. The basicness implies that the tropical map $\Sigma(\mathfrak{f}|_{t_b})$ is obtained by pulling back $\Sigma(\mathfrak{f}|_{s_b})$ along $\sigma_t \to \sigma_s$. As explained in [3, Remark 2.46], this then defines a contraction of types $\phi_{s\to t} \colon \tau_s \to \tau_t$ as in §B.3.2. In fact, one may check that $\sigma_t \to \sigma_s$ is the morphism given by (B.26). Lemma B.9 further implies that $\phi_{s\to t}(\mathbf{V}_{\max}(G_s)) \subset \phi_{s\to t}(\mathbf{V}_{\max}(G_t))$. Hence $\phi_{s\to t}$ induces a contraction of $\curlywedge$-types as in (3.7):

$$\phi_{s\to t} \colon \tau_{\curlywedge,s} = (\tau_s, \mathbf{V}_{\max}(G_s)) \to \tau_{\curlywedge,t} = (\tau_t, \mathbf{V}_{\max}(G_t)).$$

Step 4. Verifying $\tau_{\curlywedge}$-marking on the point-wise level. Next we verify that $S' = S''$ on the level of geometric points. Using the above notations, we have arrived at a commutative diagram of solid arrows of cones

$$\begin{tikzcd}
\sigma_{\curlywedge} \arrow[r, dashed] \arrow[rr, bend left] \arrow[d, hook] & \sigma_{\curlywedge,t} \arrow[r, "(3.9)"'] \arrow[d, hook] & \sigma_{\curlywedge,s} \arrow[d, hook] \\
\sigma \arrow[r] \arrow[rr, bend right] & \sigma_t \arrow[r, "(B.26)"] & \sigma_s
\end{tikzcd} \tag{3.17}$$

where the vertical arrows are inclusions of $\curlywedge$-basic cones as in §3.2.1, $\sigma_{\curlywedge} \to \sigma_{\curlywedge,s}$ is given by the $\tau_{\curlywedge}$-marking, and $\sigma \to \sigma_t$ and $\sigma \to \sigma_s$ are given by the τ-marking. We will see that the existence of a dashed arrow making (3.17) commutative leads to a unique $\tau_{\curlywedge}$-marking of the fiber $\mathfrak{f}|_t$.

Taking the dual of the top of (3.17), we obtain a commutative triangle of solid arrows

$$\begin{tikzcd}
\sigma^\vee_{\curlywedge,\mathbb{Z}} & \sigma^\vee_{\curlywedge,s,\mathbb{Z}} = \overline{\mathcal{M}}_{S,s} \arrow[l, "\chi_{\phi_s}"'] \arrow[d, "\chi_{\phi_{s\to t}}"] & \overline{\mathcal{K}}^{\curlywedge}_s \arrow[l, hook'] \arrow[d] \\
& \sigma^\vee_{\curlywedge,t,\mathbb{Z}} = \overline{\mathcal{M}}_{S,t} \arrow[ul, dashed, "\chi_{\phi_t}"] & \overline{\mathcal{K}}'_t \arrow[l, hook']
\end{tikzcd} \tag{3.18}$$

where $\overline{\mathcal{K}}'_t$ is the monoid ideal generated by the image of $\overline{\mathcal{K}}^{\curlywedge}_s$.

We first observe that the following two conditions are equivalent

(a) The image of the composition $\sigma_{\curlywedge} \to \sigma \to \sigma_t$ is contained in $\sigma_{\curlywedge,t}$.

(b) $0 \notin \overline{\mathcal{K}}'_t$ or equivalently $\overline{\mathcal{K}}'_t \subsetneq \overline{\mathcal{M}}_{S,t}$.

Both $\sigma_{\curlywedge}$ and $\sigma_{\curlywedge,t}$ can be viewed as faces of $\sigma_{\curlywedge,s}$ via the corresponding morphisms. Given (a) we observe that each element of $\overline{\mathcal{K}}'_t \subset \sigma^\vee_{\curlywedge,t,\mathbb{Z}}$ as a non-negative linear functions on $\sigma_{\curlywedge,t,\mathbb{Z}}$, restricts to a positive function in the interior of the face $\sigma_{\curlywedge} \subset \sigma_{\curlywedge,t}$. Conversely suppose that $\sigma_{\curlywedge} \cap \sigma_{\curlywedge,t}$ is a proper face of both $\sigma_{\curlywedge}$ and $\sigma_{\curlywedge,t}$. Then we may find a linear function $u \in \sigma^\vee_{\curlywedge,s,\mathbb{Z}}$ which is positive in the interior of $\sigma_{\curlywedge}$, hence $u \in \overline{\mathcal{K}}^{\curlywedge}_s$, but restricts to zero along $\sigma_{\curlywedge,t}$. This shows the equivalence of (a) and (b).

Second, we observe that (b) is equivalent to $t \in S''$. Indeed, S'' is by construction the locus along which the log ideal $\mathcal{K}^{\curlywedge}_S$ is non-trivial.

Next, we verify that (a) is equivalent to that $\mathfrak{f}_{\curlywedge}|_t$ admits a unique $\tau_{\curlywedge}$-marking induced by its τ-marking. Consequently, we have $S' = S''$ as sets of closed points.

Assuming (a), we obtain the dashed arrows in both (3.17) and (3.18) making the diagrams commutative. It suffices to show that $\phi_t(\mathbf{V}_{\max}(G_t)) \subset \mathbf{V}_{\max}(G)$. Suppose on the contrary that there exists a $V_t \in \mathbf{V}_{\max}(G_t)$ such that $\phi_t(V_t) = V \notin \mathbf{V}_{\max}(G)$. Choose a vertex $V_s \in \mathbf{V}(G_s)$ such that $\phi_{s\to t}(V_s) = V_t$. As $\phi_s(\mathbf{V}_{\max}(G_s)) \subset V_{\max}(G)$ by the $\tau_{\curlywedge}$-marking of $\mathfrak{f}_s$, we have that $V_s \notin \mathbf{V}_{\max}(G_s)$. Let

$$e_{\max,s} \in \sigma^\vee_{\curlywedge,s,\mathbb{Z}}, \qquad e_{\max,t} \in \sigma^\vee_{\curlywedge,t,\mathbb{Z}}, \qquad e_{\max} \in \sigma^\vee_{\curlywedge,\mathbb{Z}},$$

be the corresponding maximal degeneracies, and let e_{V_s}, e_{V_t}, e_V be the degeneracies of the corresponding vertices. Consider the elements in the corresponding $\curlywedge$-basic monoids

$$\delta_s := e_{\max,s} - e_{V_s} \not\succcurlyeq 0, \qquad \delta_t := e_{\max,t} - e_{V_t} = 0, \qquad \delta := e_{\max} - e_V \not\succcurlyeq 0.$$

The commutativity of (3.18) implies that

$$\delta = \chi_{\phi_s}(\delta_s) = \chi_{\phi_s} \circ \chi_{\phi_{s\to t}}(\delta_s) = \chi_{\phi_t}(\delta_t) = 0.$$

Thus $V \notin \mathbf{V}_{\max}(G)$ is maximally degenerated, a contradiction.

Step 5. Verifying $\tau_{\curlywedge}$-marking on the scheme-theoretic level. Finally, assuming $\mathfrak{f}_t$ is marked by $\tau_{\curlywedge}$, we obtain the dashed arrows in both (3.17) and (3.18) making the diagrams commutative, which gives (a). The commutativity (3.18) further implies that $\overline{\mathcal{K}}'_t$ is the $\curlywedge$-basic monoid ideal. Hence the idealized structure as in Definition 3.11 is precisely given by the vanishing $\alpha_S(\mathcal{K}^{\curlywedge}_S) = 0$ defining S'', thus verifying $S' = S''$ on the schematic level. This finishes the proof. □

The proof of the above result implies that

Corollary 3.17. *Suppose that a* $\curlywedge$*-type* $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$ *is realizable, and hence* τ *is also realizable. Let* $\sigma_{\curlywedge}$ *and* σ *be the corresponding basic cones. If the morphism of monoid ideals* $\sigma^{\vee}_{\mathbb{Z}} \setminus \{0\} \to \sigma^{\vee}_{\curlywedge,\mathbb{Z}} \setminus \{0\}$ *induced by* (3.3) *is surjective, then* (3.15) *is also an ideally strict open embedding. In particular,* $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \subset \mathfrak{U}(\mathcal{A}, \tau)$ *is a component.*

Proof. The ideal strictness of (3.15) follows from Lemma 3.9 and the definition of the τ- and the $\tau_{\curlywedge}$-marking ideals. The proof of Proposition 3.16 implies that the closed sub-stack $\mathfrak{U}(\tau_{\curlywedge}) \subset \mathfrak{U}(\tau)$, is defined precisely by the vanishing of the ideal $\alpha(\mathcal{K}_{\mathfrak{U}(\tau_{\curlywedge})}) = 0$, which is an empty condition when (3.15) is ideally strict. □

3.2.4. *Equi-dimensionality.*

Proposition 3.18. *Suppose the* $\curlywedge$*-type* $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$ *is realizable with the* $\curlywedge$*-basic cone* $\sigma_{\curlywedge}$*. Then the log stack* $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ *is non-empty, reduced and of pure dimension*

$$\dim \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) = 3 \cdot \left(\sum_{G_i \subset G} \boldsymbol{g}(G_i) - 1 \right) + |\mathbf{L}(G)| - \dim_{\mathbb{R}} \sigma_{\curlywedge}. \tag{3.19}$$

where the summation runs through connected components $G_i \subset G$*, and*

$$\boldsymbol{g}(G_i) = h^1(G_i) + \sum_{V \in \mathbf{V}(G_i)} \boldsymbol{g}(V)$$

defined as in (B.8)*.*

In the proof, we will consider the log stacks $\mathcal{A}_P$ and $\mathcal{A}_{P,I}$, defined for any fs monoid P and monoid ideal $K \subset P$, with their underlying given by the global quotients

$$\underline{\mathcal{A}}_P = \left[\operatorname{Spec} \mathbf{k}[P] / \operatorname{Spec} \mathbf{k}[P^{gp}]\right], \qquad \underline{\mathcal{A}}_{P,K} = \left[\operatorname{Spec}\left(\mathbf{k}[P]/\langle K \rangle\right) / \operatorname{Spec} \mathbf{k}[P^{gp}]\right], \tag{3.20}$$

and their log structure induced by P. Note that $\mathcal{A}_{P,I}$ is naturally equipped with a log-ideal $\mathcal{K}$ generated by K such that the strict closed embedding $\mathcal{A}_{P,K} \to \mathcal{A}_P$ is defined by the vanishing $\alpha(\mathcal{K}) = 0$. Thus, we further view $\mathcal{A}_{P,K}$ as an idealized log stack.

Proof. Consider the idealized log stack of log curves $(\mathfrak{M} := \mathfrak{M}(\mathbf{G}), \mathcal{K}^n_{\mathfrak{M}})$ from §B.1.8 with $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$. Composing (3.15), (3.13) and (B.34), we obtain an idealized log étale morphism

$$\left(\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}), \mathcal{K} := \mathcal{K}_{\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})}\right) \to (\mathfrak{M}, \mathcal{K}^n_{\mathfrak{M}}).$$

To prove the statement, we analyze this morphism locally around a geometric point $S \in \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ corresponding to a $\curlywedge$-basic punctured map $\mathfrak{f} \colon C^{\circ} \to \mathcal{A}$ over S with the basic monoid $P := \overline{\mathcal{M}}_S$.

Denote by $\tau_{\curlywedge,S} = (\tau_S, \mathbf{V}_{\max}(G_S))$ the $\curlywedge$-type of $\mathfrak{f}$ with the contraction $\phi_S \colon \tau_{\curlywedge,S} \to \tau_{\curlywedge}$. By assumption we have the $\curlywedge$-basic monoid ideal $K := \overline{\mathcal{K}}^{\curlywedge}_{\phi_S} \subset \overline{\mathcal{M}}_S$. By the construction of $\mathcal{K}_{\tau_{\curlywedge}}$ in Definition 3.11, we obtain a strict étale neighborhood $V \to \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ together with a chart of idealized log stacks

$$(V, \mathcal{K}|_V) \to \mathcal{A}_{P,K}.$$

Consider the image $S' \in \mathfrak{M}$ of $S \in \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ where $\mathcal{M}_{S'}$ is the log structure pulled back from $\mathfrak{M}$ with the decorated graph $(G_S, \boldsymbol{g}_S, \mathbf{deg}_S, \mathbf{m}_S)$. Denote by $\mathbb{N}^n$ the free monoid generated by edge lengths of $\mathbf{E}(G_S)$ with $n = |\mathbf{E}(G_S)|$, and let $K^n \subset \mathbb{N}^n$ be the monoid ideal generated by edge lengths corresponding to edges in $\mathbf{E}(\phi_S)$. By the construction of nodal ideals in §B.1.8, we obtain a strict étale neighborhood $V' \to \mathfrak{M}$ of S' together with a chart of idealized log stacks

$$(V', \mathcal{K}_n|_{V'}) \to \mathcal{A}_{\mathbb{N}^n, K^n}.$$

By [3, Proposition B.2], we may choose appropriate V and V' to fit in a commutative diagram

$$\begin{tikzcd}
(V, \mathcal{K}|_V) \arrow[r] \arrow[d] & \mathcal{A}_{P,K} \arrow[d] \\
(V', \mathcal{K}_n|_{V'}) \arrow[r] & \mathcal{A}_{\mathbb{N}^n, K^n}
\end{tikzcd} \tag{3.21}$$

where the horizontal arrows are given by the above charts, and the right vertical arrow is a consequence of basicness. Further applying [3, Proposition B.4], we obtain a strict étale morphism of idealized log stacks

$$(V, \mathcal{K}|_V) \to (V', \mathcal{K}_n|_{V'}) \times_{\mathcal{A}_{\mathbb{N}^n, K^n}} \mathcal{A}_{P,K}. \tag{3.22}$$

The idealized log smoothness of $(\mathfrak{M}, \mathcal{K}^n_{\mathfrak{M}})$ implies that the underlying of the bottom arrow in the diagram is smooth. In particular, the neighborhood V' is reduced. To prove that V and hence $\mathfrak{U}(\mathcal{A}, \tau_\curlywedge)$ is reduced, it suffices to observe that $\mathcal{A}_{P,K}$ is reduced. Since $\mathcal{A}_{P,K} \subset \mathcal{A}_P$ is defined by the monoid ideal K, it suffices to show that $\langle K \rangle \subset \mathbf{k}[P]$ has no nilpotents. This follows from the definition of $\curlywedge$-basic monoid ideals in §3.2.1.

Next we observe that $\mathfrak{U}(\mathcal{A}, \tau_\curlywedge)$ is non-empty. By the realizability of $\tau_\curlywedge$, we have a tropical map $f^{\mathrm{trop}} \colon \Gamma(G, \ell) \to \mathbb{R}_{\geq 0}$ over the $\curlywedge$-basic cone $\sigma_\curlywedge$ of type $\tau_\curlywedge$. Repeating the construction in [3, Prop. 3.29], we obtain a punctured curve $C^\circ \to S$ over a geometric log point S with $\overline{\mathcal{M}}_S = \sigma^\vee_{\curlywedge, \mathbb{Z}}$, such that the corresponding tropical curve is given by the type $\tau_\curlywedge$. Since $\mathcal{A}$ has a Zariski log structure, [2, Prop. 2.10] produces a punctured map $\mathfrak{f} \colon C^\circ \to \mathcal{A}$ over S, whose tropicalization is f^{trop}, hence of type $\tau_\curlywedge$ as needed.

We use (3.22) to compute the dimension of $\mathfrak{U}(\mathcal{A}, \tau_\curlywedge)$. We may choose $\mathfrak{f}$ to be a $\curlywedge$-basic map of type $\tau_\curlywedge$ with the $\curlywedge$-basic monoid ideal $K = \sigma^\vee_{\curlywedge, \mathbb{Z}} \setminus \{0\}$. In this case $n = |\mathbf{E}(G)|$, and $K^n = \mathbb{N}^n \setminus \{0\}$. Then we have

$$\dim \mathcal{A}_{P,K} = -\dim \sigma_\curlywedge, \qquad \dim \mathcal{A}_{\mathbb{N}^n, K} = -|\mathbf{E}(G)|.$$

Finally we compute

$$\begin{aligned}
\dim \mathfrak{U}(\mathcal{A}, \tau_\curlywedge) &= \dim \mathfrak{M} + \dim \mathcal{A}_{P,K} - \dim \mathcal{A}_{\mathbb{N}^n, K} \\
&= 3 \cdot \left(\sum_{G_i \subset G} \boldsymbol{g}(G_i) - 1 \right) + |\mathbf{L}(G)| - |\mathbf{E}(G)| - \dim \sigma_\curlywedge - (-|\mathbf{E}(G)|) \\
&= 3 \cdot \left(\sum_{G_i \subset G} \boldsymbol{g}(G_i) - 1 \right) + |\mathbf{L}(G)| - \dim \sigma_\curlywedge,
\end{aligned}$$

as desired. □

Remark 3.19. The proof of the above proposition shows that $\mathfrak{U}(\mathcal{A}, \tau_\curlywedge)$ contains an open dense substack consisting of punctured maps of $\curlywedge$-type $\tau_\curlywedge$.

3.2.5. *The maximal degeneracy and the class $\psi_{\max}$.* Let $\mathcal{A}_{\max} := \mathcal{A}$, and let $\Delta_{\max} \subset \mathcal{A}_{\max}$ be the strict closed substack given by the origin. Let $e_{\max} \in \Gamma(\mathcal{A}_{\max}, \overline{\mathcal{M}}_{\mathcal{A}_{\max}}) \cong \mathbb{N}$ be the generator. Then we may view $\Delta_{\max}$ as an idealized stack with the log-ideal $\mathcal{K}_{\max} = \mathcal{M}_{\Delta_{\max}} \setminus \mathcal{O}^*$ generated by the preimage of $e_{\max}$.

For a $\curlywedge$-type $\tau_\curlywedge = (\tau, \mathbf{V}_{\max}(G))$, consider the corresponding maximal degeneracies

$$e_{\max}(\tau_\curlywedge) \in \Gamma(\mathfrak{U}(\mathcal{A}, \tau_\curlywedge), \overline{\mathcal{M}}_{\mathfrak{U}(\mathcal{A}, \tau_\curlywedge)}), \qquad e_{\max}(\tau) \in \Gamma(\mathfrak{U}(\mathcal{A}, \tau), \overline{\mathcal{M}}_{\mathfrak{U}(\mathcal{A}, \tau)}).$$

By (A.6), these global sections induce canonical morphisms

$$\mathfrak{U}(\mathcal{A}, \tau_\curlywedge) \to \mathcal{A}_{\max}, \qquad \mathfrak{U}(\mathcal{A}, \tau) \to \mathcal{A}_{\max}. \tag{3.23}$$

We will use $\psi_{\max}$ to denote the class $-c_1(\Delta_{\max})$ and its pull-backs. We may also use the notation $\mathbf{L}_{\max}$ to denote the line bundle $\mathcal{O}(-\Delta_{\max})$ and its pull-backs when there is no danger of confusion.

3.3. The $\curlywedge$-decomposition of log maps to $\mathcal{A}$.

3.3.1. *The universal stack of log maps and its boundary.* We fix a type

$$\tau^\star = (G^\star, \mathbf{g}, \mathbf{deg}, \mathbf{m}, \boldsymbol{\sigma}, \mathbf{c}), \tag{3.24}$$

where $\mathbf{V}(G^\star) = \{\star\}$, $\mathbf{E}(G) = \emptyset$, $\boldsymbol{\sigma}(\star) = 0$, and $\mathbf{c}(L) \in \mathbb{N}$ for any $L \in \mathbf{L}(G^\star)$. It induces a unique $\curlywedge$-type $\tau^\star_\curlywedge = (\tau^\star, \mathbf{V}_{\max}(G^\star) = \{\star\})$ since $\mathbf{V}_{\max}(G^\star) \neq \emptyset$. Note that both $\tau^\star$ and $\tau^\star_\curlywedge$ are realizable. Furthermore, in this case the basic and the $\curlywedge$-basic cones are identical, and are the trivial cone $\{0\}$. By Corollary 3.17, we have isomorphims

$$\mathfrak{U}(\mathcal{A}, \tau^\star_\curlywedge) \cong \mathfrak{U}(\mathcal{A}, \tau^\star) \cong \mathfrak{U}'(\mathcal{A}, \tau^\star), \qquad \mathfrak{M}(\mathcal{A}, \tau^\star) \cong \mathfrak{M}'(\mathcal{A}, \tau^\star).$$

In this case, the tautological morphism $\mathfrak{U}(\mathcal{A}, \tau^\star) \to \mathfrak{M}(\mathcal{A}, \tau^\star)$ as in (3.12) is proper, log étale, and birational. Indeed, it restricts to an isomorphism on the open dense substacks with the trivial log structures, see [20, Thm 3.17].

Lemma 3.20. *The tautological morphism of log stacks*

$$\mathfrak{U}(\mathcal{A}, \tau^\star) \longrightarrow \mathfrak{M}(\mathcal{A}, \tau^\star) \times \mathcal{A}_{\max}$$

induced by (3.12) *and* (3.23) *is log étale.*

Proof. We will construct a dashed arrow making the following diagram of solid arrows commutative:

$$\begin{tikzcd}
T_0 \arrow[r] \arrow[d] & \mathfrak{U}(\mathcal{A}, \tau^\star) \arrow[d] \\
T \arrow[r] \arrow[ur, dashed] & \mathfrak{M}(\mathcal{A}, \tau^\star) \times \mathcal{A}_{\max}
\end{tikzcd}$$

where the left vertical arrow is strict closed immersion defined by a square zero ideal. This diagram is equivalent to the following diagram

$$\begin{tikzcd}
C_0 \arrow[r] \arrow[d] \arrow[rr, bend left, "{(f_0, e_{\max,T_0})}"] & C \arrow[r, dashed] \arrow[d] & \mathcal{A} \times \mathcal{A}_{\max} \arrow[d] \\
T_0 \arrow[r] & T \arrow[r, "e_{\max,T}"] & \mathcal{A}_{\max}
\end{tikzcd}$$

where the dashed arrow in the second diagram corresponds to the dashed arrow in the first diagram. Since $C_0 \to C$ is a strict infinitesimal thickening, and $\mathcal{A} \times \mathcal{A}_{\max} \to \mathcal{A}_{\max}$ is log étale, we observe that the dashed arrow in the second, and hence in the first diagram exists, and is unique, and makes the whole diagrams commutative. This finishes the proof. ☐

By the above lemma, the canonical morphism $\Phi_{\max}\colon \mathfrak{U}(\mathcal{A}, \tau^\star) \to \mathcal{A}_{\max}$ as in (3.23) is log smooth. Consider the boundary log stack

$$\Delta^\curlywedge(\tau^\star) := \mathfrak{U}(\mathcal{A}, \tau^\star) \times_{\mathcal{A}_{\max}} \Delta_{\max} \tag{3.25}$$

with its idealized structure $\mathcal{K}_{\Delta^\curlywedge_{g,\mathbf{c}}(\mathcal{A})}$ defined by pulling back the log ideal $\mathcal{K}_{\max}$ over $\Delta_{\max}$. The projection

$$\Phi_{\Delta_{\max}}\colon (\Delta^\curlywedge(\tau^\star), \mathcal{K}_{\Delta^\curlywedge(\tau^\star)}) \longrightarrow (\Delta_{\max}, \mathcal{K}_{\max})$$

is thus log smooth and ideally strict, hence is idealized log smooth. Proposition 3.18 implies that $\Delta^\curlywedge(\tau^\star)$ is equidimensional with

$$\dim \Delta^\curlywedge(\tau^\star) = 3g - 4 + |\mathbf{L}(G)|,$$

where $g := \mathbf{g}(\star)$. Denote by $[\Delta^\curlywedge(\tau^\star)]$ the corresponding fundamental class.

3.3.2. *Decomposition of* $[\Delta^{\curlywedge}(\tau^{\star})]$. Denote by $\Sigma(\Delta^{\curlywedge}(\tau^{\star}))$ the generalized cone complex of $\Delta^{\curlywedge}(\tau^{\star})$. We observe that the tropicalization of $\Phi_{\Delta_{\max}}$ is

$$\Sigma(\Phi_{\Delta_{\max}})\colon \Sigma(\Delta^{\curlywedge}(\tau^{\star})) \to \Sigma(\Delta_{\max}) = \mathbb{R}_{\geq 0}.$$

Indeed, consider a strict geometric log point $S \in \Delta^{\curlywedge}(\tau^{\star})$ corresponds to a log map $\mathfrak{f}\colon C \to \mathcal{A}$ with the $\curlywedge$-type $\tau_{\curlywedge}$ and the $\curlywedge$-basic cone $\sigma_{\curlywedge}$. Then the restriction of $\Sigma(\Phi^{\mathrm{trop}}_{\Delta_{\max}})$ to the cone $\sigma_{\curlywedge} \in \Sigma(\Delta^{\curlywedge}(\tau^{\star}))$ is the linear map given by the maximal degeneracy (3.4).

By the log smoothness of $\Phi_{\max}$ and [2, Prop. 3.1], the irreducible components of $\Delta^{\curlywedge}(\tau^{\star})$ are Weil divisors in $\mathfrak{U}(\mathcal{A}, \tau^{\star})$, and are classified by the one-dimensional cones of $\Sigma(\Delta^{\curlywedge}(\tau^{\star}))$ which surject onto $\mathbb{R}_{\geq 0}$ under $\Sigma(\Phi_{\Delta_{\max}})$. Let $\rho \in \Sigma^1(\Delta^{\curlywedge}(\tau^{\star}))$ be a 1-dimensional cone, and $\Delta_{\rho} \subset \Delta^{\curlywedge}(\tau^{\star})$ be the corresponding reduced irreducible component. The dual of the restriction $\Sigma(\Phi_{\Delta_{\max}})|_{\rho}$ yields a morphism of monoids

$$\Sigma(\Phi_{\Delta_{\max}})|_{\rho}^{\vee}\colon \mathbb{N} \to \rho_{\mathbb{Z}}^{\vee} \tag{3.26}$$

with image $\Sigma(\Phi_{\Delta_{\max}})|_{\rho}^{\vee} = e_{\max}|_{\rho}$ the maximal degeneracy of the tropical map parameterized by ρ. Since $\rho_{\mathbb{Z}}^{\vee} \cong \mathbb{N}$, we may view $e_{\max}|_{\rho} \in \rho_{\mathbb{Z}} \cong \mathbb{N}$ as a positive integer. By [2, Corollary 3.2], we have a decomposition of the fundamental class

$$[\Delta^{\curlywedge}(\tau^{\star})] = \sum_{\rho} e_{\max}|_{\rho} \cdot [\Delta_{\rho}]. \tag{3.27}$$

3.3.3. *Tropical refinement.* We further refine the decomposition (3.27) by classifying the irreducible components of $\Delta^{\curlywedge}(\tau^{\star})$ tropically.

For a ray $\rho \in \Sigma^1(\Delta^{\curlywedge}(\tau^{\star}))$, let $S \in \Delta_{\rho}$ be any strict geometric point corresponding to a log map $\mathfrak{f}\colon C \to \mathcal{A}$ of $\curlywedge$-type $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$. Denote by $\Sigma(\mathfrak{f})$ the corresponding tropical map over its $\curlywedge$-basic cone $\sigma_{\curlywedge}$. Note that $\sigma_{\curlywedge}$ contains ρ as a ray, see (3.9).

Now choose $S \in \Delta_{\rho}$ to be a general point. Since $\Delta_{\rho} \subset \mathfrak{U}(\tau^{\star})$ is a Weil divisor, we have $\sigma_{\curlywedge} \cong \rho$. Thus tropical curve $\Sigma(\mathfrak{f})$ over $\sigma_{\curlywedge} \cong \rho$ is rigid as in §3.1.3 with $e_{\max}(\tau_{\curlywedge}) = e_{\max}|_{\rho}$.

For a type $\tau = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}, \boldsymbol{\sigma}, \mathbf{c})$, denote by $\mathrm{Aut}(\tau)$ the automorphisms of G that fix $\mathbf{m}$ and commute with $\mathbf{g}, \mathbf{deg}, \boldsymbol{\sigma}, \mathbf{c}$. For a $\curlywedge$-type $\tau_{\curlywedge} = (\tau, \mathbf{V}_{\max}(G))$, denote by $\mathrm{Aut}(\tau_{\curlywedge}) \subset \mathrm{Aut}(\tau)$ the set of automorphisms fixing $\mathbf{V}_{\max}(G)$.

Lemma 3.21. *Suppose $\tau_{\curlywedge}$ is the $\curlywedge$-type corresponding to a general strict geometric point $S \in \Delta_{\rho}$. Then the tautological morphism $i_{\tau_{\curlywedge}}\colon \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \to \mathfrak{U}(\mathcal{A}, \tau^{\star})$ is finite and unramified. Furthermore, it is of degree $|\mathrm{Aut}(\tau_{\curlywedge})|$ onto its image Δ_{ρ}, hence*

$$[\Delta_{\rho}] = \frac{i_{\tau_{\curlywedge},*}[\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})]}{|\mathrm{Aut}(\tau_{\curlywedge})|}. \tag{3.28}$$

Proof. Consider the sequence of strict tautological morphisms

$$\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \to \mathfrak{U}(\mathcal{A}, \tau) \cong \mathfrak{U}(\mathcal{A}, \tau^{\star}) \times_{\mathfrak{M}(\mathcal{A}, \tau^{\star})} \mathfrak{M}(\mathcal{A}, \tau) \to \mathfrak{U}(\mathcal{A}, \tau^{\star}) \times_{\mathfrak{M}_{g,\mathbf{deg}}} \mathfrak{M}(\mathbf{G}),$$

where $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ be the decorated graph of $\tau_{\curlywedge}$, and $\mathfrak{M}(g, \mathbf{deg}, \mathbf{m})$ is the stack of twisted pre-stable curves with orbifold structure along makings specified by $\mathbf{deg}$. By Proposition 3.16, the left arrow is a closed embedding. The right arrow is obtained by imposing the conditions from $\boldsymbol{\sigma}$ and $\mathbf{c}$. As both conditions are closed conditions, the right arrow is again a closed embedding. Thus the composition is a closed embedding. Since the morphism $\mathfrak{M}(\mathbf{G}) \to \mathfrak{M}(g, \mathbf{deg}, \mathbf{m})$ is finite and unramified, the morphism $i_{\tau_{\curlywedge}}$ is also finite and unramified.

By Remark 3.19, the stack $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ has an open dense locus consisting of maps marked by $\tau_{\curlywedge}$. As a general object of Δ_{ρ} has $\curlywedge$-type $\tau_{\curlywedge}$, this open dense locus of $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ dominates Δ_{ρ} via i_{τ}. Hence $i_{\tau_{\curlywedge}}$ surjects onto its image Δ_{ρ}.

Consider a general strict geometric point $S \in \Delta_{\rho}$ corresponding to a log map $\mathfrak{f}\colon C \to \mathcal{A}$. It is of $\curlywedge$-type $\tau_{\curlywedge}$. The fiber $i_{\tau_{\curlywedge}}^{-1}(S)$ consists of log maps isomorphic to $\mathfrak{f}$ but with different labelings by $\tau_\curlywedge$. Thus the group $\mathrm{Aut}(\tau_\curlywedge)$ acts freely and transitively on $i^{-1}_{\tau_\curlywedge}(S)$. This finishes the proof. $\square$

Proposition 3.22. *We have the following decomposition of fundamental classes*

$$[\Delta^{\curlywedge}(\tau^\star)] = \sum_{\tau_\curlywedge \vdash \tau^\star} \frac{\mathrm{lcm}_{x\in \mathbf{E}(G)}(\mathbf{c}(x))}{|\,\mathrm{Aut}(\tau_\curlywedge)|} i_{\tau_\curlywedge,*}[\mathfrak{U}(\mathcal{A},\tau_\curlywedge)], \tag{3.29}$$

where the summation runs through all contractions $\tau_\curlywedge \vdash \tau^\star$ of $\curlywedge$-types of degenerate rigid tropical curves, and G is the graph of $\tau_\curlywedge$.

Proof. Note that the open dense substack of $\mathring{\mathfrak{M}}(\mathcal{A},\tau^\star) \subset \mathfrak{M}(\mathcal{A},\tau^\star)$ with the trivial log structure admits the type $\tau^\star$. Therefore, any log map parameterized by $\Delta^{\curlywedge}(\tau^\star)$ admits a smoothing to $\mathring{\mathfrak{M}}(\mathcal{A},\tau^\star)$. These smoothings on the combinatorial side correspond to contractions to $\tau^\star$. Hence $\tau_\curlywedge$ in the summation admits contractions to $\tau^\star$, as required. Thus, the formula follows from (3.6), (3.27) and (3.28). $\square$

4. THE REDUCED THEORY AND THE TROPICAL DECOMPOSITION FORMULA

Throughout this section, we assume that $\mathcal{X}$ is smooth as in Assumption 2.7. We will extend the construction in [18] to the punctured situation to construct a reduction of the canonical perfect obstruction theory of punctured R-maps, as constructed in §2.6.

4.1. Superpotentials.

Recall from [18, §3.4] that a *superpotential* is a morphism

$$W\colon \mathfrak{P}^\circ \to \mathcal{L}_\omega \tag{4.1}$$

over $\mathbf{BC}^*_\omega$. The choice of a superpotential is a key ingredient in the reduction procedure. Differentiating W, we obtain a morphism of tangent bundles relative to $\mathbf{BC}^*_\omega$:

$$\mathrm{d}\,W\colon T_{\mathfrak{P}^\circ/\mathbf{BC}^*_\omega} \to W^* T_{\mathcal{L}_\omega/\mathbf{BC}^*_\omega} \cong W^*\mathcal{L}_\omega.$$

The *critical locus* $\mathrm{Crit}(W)$ of W is defined to be closed substack of W where $\mathrm{d}\,W$ degenerates. We say that W has *proper critical locus* if $\mathrm{Crit}(W)$ is proper over $\mathbf{BC}^*_\omega$.

Remark 4.1. We may view W as a section of the line bundle $\mathcal{L}_\omega|_{\mathfrak{P}^\circ}$, hence a rational section of $\mathcal{L}_\omega|_{\mathfrak{P}}$. By [18, Lemma 3.8], the existence of a non-zero superpotential implies that the rational number $\tilde{r} = a\cdot r$ as in §2.1.1 is the pole order of W along $\infty \subset \mathfrak{P}$, and hence an integer. Thus, the existence of a non-zero superpotential imposes a non-trivial constraint on targets.

A non-zero superpotential W does not extend to a holomorphic function over $\mathfrak{P}$. To fix this issue, we further modify W as follows. Consider $\mathbb{P}_\omega = \mathbb{P}(\mathcal{L}_\omega \oplus \mathcal{O}) \to \mathbf{BC}^*_\omega$ where we equip $\mathbb{P}_\omega$ with the divisorial log structure given by the smooth divisor $\infty_{\mathbb{P}_\omega} := \mathbb{P}_\omega \setminus \mathcal{L}_\omega$. In particular $\mathbb{P}_\omega$ is of DF1, and admits a canonical strict morphism $\mathbb{P}_\omega \to \mathcal{A}$.

Consider the log étale morphism of log stacks

$$\mathcal{A}^e \to \mathcal{A}\times\mathcal{A} \tag{4.2}$$

given by blowing up the origin of $\mathcal{A}\times\mathcal{A}$. Let $\mathfrak{P}^e$ and $\mathbb{P}^e_\omega$ be the pull-back of (4.2) along

$$\mathfrak{P}\times\mathcal{A}_{\max} \to \mathcal{A}\times\mathcal{A}, \qquad \mathbb{P}_\omega\times\mathcal{A}_{\max} \to \mathcal{A}\times\mathcal{A}$$

respectively, where the morphisms $\mathfrak{P}\to\mathcal{A}$ and $\mathbb{P}_\omega \to \mathcal{A}$ are the strict canonical ones, the morphism $\mathcal{A}_{\max}\to\mathcal{A}$ on the left is the identity, and $\mathcal{A}_{\max}\to\mathcal{A}$ on the right is the degree $\tilde{r}$ morphism defined by $\tilde{r} \mapsfrom 1$ on the level of characteristics.

Denote by $\infty_{\mathfrak{P}^e} \subset \mathfrak{P}^e$ and $\infty_{\mathbb{P}^e_\omega} \subset \mathbb{P}^e_\omega$ the proper transform of $\infty_{\mathfrak{P}}\times\mathcal{A}_{\max}$ and $\infty_{\mathbb{P}_\omega\times\mathcal{A}_{\max}}$ respectively. Consider their complements

$$\mathfrak{P}^{e,\circ} := \mathfrak{P}^e \setminus \infty_{\mathfrak{P}^e}, \qquad \mathbb{P}^{e,\circ}_\omega := \mathbb{P}^e_\omega \setminus \infty_{\mathbb{P}^e_\omega}.$$

We obtain the following commutative diagram

$$\begin{tikzcd}
\mathfrak{P}^{e,\circ} \arrow[r, dashed, "W^{e,\circ}"] \arrow[d] & \mathbb{P}^{e,\circ}_{\omega} \arrow[r, "\mathfrak{c}"] \arrow[d] & \mathcal{V} := \mathrm{Vb}(\mathcal{L}_{\omega} \boxtimes \mathcal{O}_{\mathcal{A}_{\max}}(\tilde{r}\Delta_{\max})) \arrow[d] \\
\mathfrak{P} \times \mathcal{A}_{\max} \arrow[r, dashed, "W \times id_{\mathcal{A}_{\max}}"] & \mathbb{P}_{\omega} \times \mathcal{A}_{\max} \arrow[r] & \mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}
\end{tikzcd} \tag{4.3}$$

where the bottom dashed arrow is the rational map defined as the product of W and the identity of $\mathcal{A}_{\max}$, the top horizontal dashed arrow is the induced rational map as both vertical arrows are birational, and the morphism $\mathfrak{c}$ is the contraction of the proper transform of $\mathbb{P}_{\omega} \times \Delta_{\max}$. We may equip $\mathrm{Vb}(\mathcal{L}_{\omega} \boxtimes \mathcal{O}_{\mathcal{A}_{\max}}(\tilde{r}\Delta_{\max}))$ with the log structure pulled back from $\mathcal{A}_{\max}$.

As shown in [18, Prop. 3.10], the composition

$$\widetilde{W} := \mathfrak{c} \circ W^{e,\circ} \colon \mathfrak{P}^{e,\circ} \longrightarrow \mathcal{V}, \tag{4.4}$$

called the *twisted superpotential*, is a surjective morphism of log stacks over $\mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}$, contracting the proper transform $\widehat{\mathfrak{P}} \subset \mathfrak{P}^{e,\circ}$ of $\mathfrak{P} \times \Delta_{\max}$. Taking differentials, we obtain a morphism of log tangent bundles

$$\mathrm{d}\,\widetilde{W} \colon T_{\mathfrak{P}^{e,\circ}/\mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}} \to \widetilde{W}^{*} T_{\mathcal{V}/\mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}} \cong \widetilde{W}^{*}\left(\mathcal{L}_{\omega} \boxtimes \mathcal{O}_{\mathcal{A}_{\max}}(\tilde{r}\Delta_{\max})\right).$$

Note that the left vertical arrow in (4.3) is given by the base change of (4.2), hence is log étale. This yields an isomorphism $T_{\mathfrak{P}^{e,\circ}/\mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}} \cong T_{\mathfrak{P}/\mathbf{BC}^{*}_{\omega}}|_{\mathfrak{P}^{e,\circ}}$, and thus we obtain the morphism

$$\mathrm{d}\,\widetilde{W} \colon T_{\mathfrak{P}/\mathbf{BC}^{*}_{\omega}}|_{\mathfrak{P}^{e,\circ}} \to \widetilde{W}^{*}\left(\mathcal{L}_{\omega} \boxtimes \mathcal{O}_{\mathcal{A}_{\max}}(\tilde{r}\Delta_{\max})\right). \tag{4.5}$$

The *critical locus* of $\widetilde{W}$ is defined to be the closed substack $\mathrm{Crit}(\widetilde{W}) \subset \mathfrak{P}^{e,\circ}$ defined by the vanishing of $\mathrm{d}\,\widetilde{W}$. By [18, Prop. 3.12], the substack $\mathrm{Crit}(\widetilde{W})$ is proper over $\mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}$ if and only if W has proper critical locus.

4.2. **Set-up of the reduction.** Consider a decorated type as in (2.18):

$$\boldsymbol{\tau} = (\tau, \overline{\gamma}, \boldsymbol{\beta}), \qquad \text{with } \tau = (\mathbf{G}, \boldsymbol{\sigma}, \mathbf{c}) \text{ and } \mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}). \tag{4.6}$$

We further assume that $\boldsymbol{\tau}$ is of compact type, see §2.3.1. Let $\mathbf{L}_0(G) \subset \mathbf{L}(G)$ be the subset of legs with the zero contact order, and $\mathbf{S}(G) \subset \mathbf{E}(G)$ be the set of edges with non-zero contact orders. Since $\boldsymbol{\tau}$ is of compact type, the set $\mathbf{L}_{\infty}(G) := \mathbf{L}(G) \setminus \mathbf{L}_0(G)$ consists of legs with strictly negative contact orders. Furthermore, for $x = \{h, \hat{h} = \iota_G(h)\} \in \mathbf{E}(G)$ with $\mathbf{c}(x) = 0$ we require the nodal sectors $\overline{\gamma}_h, \overline{\gamma}_{\hat{h}}$ to be a pair of $\mathfrak{P}$-sectors.

Let $G = \sqcup_i G_i$ be the union of connected components. For later use, denote by τ_i the type by restricting τ to G_i.

Recall the evaluation stacks $\mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau)$, $\mathfrak{M}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau)$ from §2.6.3. From here, we may define $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau)$ and $\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau)$ to fit into the following commutative diagram with Cartesian squares and strict vertical arrows:

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}) \arrow[r, "\mathbf{ev}"] \arrow[d] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau) \arrow[r] \arrow[d] & \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau) \arrow[r] \arrow[d] & \mathfrak{U}(\mathcal{A}, \tau) \arrow[d, "(3.12)"] \\
\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}) \arrow[r, "(2.54)"] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau) \arrow[r] & \mathfrak{M}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau) \arrow[r] & \mathfrak{M}(\mathcal{A}, \tau)
\end{tikzcd} \tag{4.7}$$

For simplicity, we write

$$\mathscr{U} := \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}), \qquad \mathfrak{U}^{\mathrm{ev}} := \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau), \qquad \mathfrak{U} := \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau), \tag{4.8}$$

and denote the universal family over $\mathscr{U}$ by

$$f_{\mathscr{U}} \colon C^{\circ}_{\mathscr{U}} \to \mathfrak{P}, \qquad \pi_{\mathscr{U}} \colon C^{\circ}_{\mathscr{U}} \to \mathscr{U} \tag{4.9}$$

Let $\widetilde{C^\circ_{\mathscr{U}}} \to C^\circ_{\mathscr{U}}$ be the partial normalization along nodal gerbes in $\mathbf{S}$. Composing with the the arrows in (4.9), we obtain the corresponding arrows

$$\tilde{f}_{\mathscr{U}} \colon \widetilde{C^\circ_{\mathscr{U}}} \to \mathfrak{P}, \qquad \tilde{\pi}_{\mathscr{U}} \colon \widetilde{C^\circ_{\mathscr{U}}} \to \mathscr{U}. \tag{4.10}$$

Let $p_x \subset C^\circ_{\mathscr{U}}$ be the gerbe corresponding to $x \in \mathbf{L}(G) \cup \mathbf{S}$. For an edge $x = \{h, \hat{h}\} \in \mathbf{S}$, denote by $p_h, p_{\hat{h}} \subset \widetilde{C^\circ_{\mathscr{U}}}$ the pre-images of p_x. For later use, write

$$\Sigma_0 := \sum_{x \in \mathbf{L}_0(G)} p_x, \qquad \Sigma_\infty = \sum_{x \in \mathbf{L}_\infty(G)} p_x, \qquad \widetilde{\Sigma} = \sum_{x \in \mathbf{L}(G)} p_x + \sum_{\{h,\hat{h}\} \in \mathbf{S}} (p_h + p_{\hat{h}}).$$

By (2.56), we obtain compatible perfect obstruction theories

$$\begin{aligned} \varphi_{\boldsymbol{\tau}} \colon \mathbb{T}_{\mathscr{U}/\mathfrak{U}} &\longrightarrow \mathbb{E}_{\mathscr{U}/\mathfrak{U}} := R\pi_{\mathscr{U},*}\left(f^*_{\mathscr{U}} T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0)\right), \\ \varphi_{\boldsymbol{\tau},\mathrm{ev}} \colon \mathbb{T}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} &\longrightarrow \mathbb{E}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} := R\tilde{\pi}_{\mathscr{U},*}\left(\tilde{f}^*_{\mathfrak{M}} T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\widetilde{\Sigma})\right). \end{aligned} \tag{4.11}$$

Fix a superpotential W as in (4.1) with proper critical locus. We will first construct a reduction for $\varphi_{\boldsymbol{\tau}}$, then deduce a compatible reduction for $\varphi_{\boldsymbol{\tau},\mathrm{ev}}$.

Remark 4.2. The construction of the reduction in this section applies identically to the case of stable punctured R-maps weakly marked by τ. However, in this paper, we do not need the reduced theory of weakly marked punctured R-maps.

4.3. **The canonical cosection.**

4.3.1. *Modifying the target.* The universal punctured R-map $f_{\mathscr{U}}$ in (4.9) is equivalent to a section

$$f_{\mathscr{U},\mathcal{P}} \colon C^\circ_{\mathscr{U}} \to \mathcal{P}_{\mathscr{U}} := \mathfrak{P} \times_{\mathbf{BC}^*_\omega} C^\circ_{\mathscr{U}} \tag{4.12}$$

of the projection $\mathcal{P}_{\mathscr{U}} \to C^\circ_{\mathscr{U}}$. Denote by $\mathfrak{X}_{\mathscr{U}} := C^\circ_{\mathscr{U}} \times_{\mathbf{BC}^*_\omega} \mathfrak{X}$. Define the log stack $\mathcal{P}_{\mathscr{U},-}$ with the underlying

$$\underline{\mathcal{P}_{\mathscr{U},-}} := \mathbb{P}^{\mathbf{w}}\left(\bigoplus_{i>0}(\mathbf{E}^\vee_{i,\mathfrak{X}_{\mathscr{U}}} \otimes \mathcal{L}_{\mathfrak{X}}^{\otimes i}|_{\mathfrak{X}_{\mathscr{U}}}(-\Sigma_0)) \oplus \mathcal{O}_{\mathfrak{X}_{\mathscr{U}}}\right)$$

where $\mathbf{w}$ are the weights in (2.2). Let $\mathcal{M}_{\infty_{\mathcal{P}_{\mathscr{U},-}}}$ be the DF1 log structure on $\underline{\mathcal{P}_{\mathscr{U},-}}$ given by the smooth divisor $\infty_{\mathcal{P}_{\mathscr{U},-}}$ defined by the vanishing of the $\mathcal{O}_{\mathfrak{X}_{\mathscr{U}}}$-coordinate. Define the log stack $\mathcal{P}_{\mathscr{U},-} = (\underline{\mathcal{P}_{\mathscr{U},-}}, \mathcal{M}_{\mathcal{P}_{\mathscr{U},-}} = \mathcal{M}_{C^\circ_\tau}|_{\mathcal{P}_{\mathscr{U},-}} \oplus_{\mathcal{O}^*} \mathcal{M}_{\infty_{\mathcal{P}_{\mathscr{U},-}}})$. Comparing with (2.2), we obtain a birational map of log stacks over $\mathfrak{X}_{\mathscr{U}}$

$$\mathcal{P}_{\mathscr{U},-} \dashrightarrow \mathcal{P}_{\mathscr{U}},$$

with indeterminancy locus $\infty_{\mathcal{P}_{\mathscr{U},-}} \times_{C^\circ_{\mathscr{U}}} \Sigma_0$. Furthermore the morphism of log stacks

$$\mathcal{P}_{\mathscr{U},reg} := \mathcal{P}_{\mathscr{U},-} \setminus (\infty_{\mathcal{P}_{\mathscr{U},-}} \times_{C^\circ_{\mathscr{U}}} \Sigma_0) \to \mathcal{P}_{\mathscr{U}} \tag{4.13}$$

contracts precisely the fiber over $\mathfrak{X}_{\mathscr{U}} \times_{C^\circ_{\mathscr{U}}} \Sigma_0$ to the zero section of $\mathfrak{P}_{\mathscr{U}}$. Set

$$\infty_{\mathcal{P}_{\mathscr{U},reg}} := \infty_{\mathcal{P}_{\mathscr{U},-}} \setminus (\infty_{\mathcal{P}_{\mathscr{U},-}} \times_{C^\circ_{\mathscr{U}}} \Sigma_0).$$

Consider the two morphisms

$$\mathcal{P}_{\mathscr{U},reg} \to \mathcal{A} \times \mathcal{A}_{\max}, \qquad \mathcal{P}_{\mathscr{U}} \to \mathcal{A} \times \mathcal{A}_{\max} \tag{4.14}$$

where the arrows to $\mathcal{A}$ are induced by the corresponding divisors $\infty_{\mathcal{P}_{\mathscr{U},reg}}$ and $\infty_{\mathcal{P}_{\mathscr{U}}}$. We construct the commutative diagram

$$\begin{tikzcd} \mathcal{P}^{e,\circ}_{\mathscr{U},reg} \arrow[r] \arrow[d] & \mathcal{P}^{e}_{\mathscr{U},reg} \arrow[r] \arrow[d] & \mathcal{P}_{\mathscr{U},reg} \arrow[d] \\ \mathcal{P}^{e,\circ}_{\mathscr{U}} \arrow[r] & \mathcal{P}^{e}_{\mathscr{U}} \arrow[r] & \mathcal{P}_{\mathscr{U}} \end{tikzcd} \tag{4.15}$$

where the two right horizontal arrows are obtained by pulling back (4.2) along (4.14), and the two left horizontal arrows are obtained by removing the proper transforms of $\infty_{\mathcal{P}_{\mathscr{U},reg}} \times \mathcal{A}_{\max}$ and $\infty_{\mathcal{P}_{\mathscr{U}}} \times \mathcal{A}_{\max}$ respectively. Note that the vertical morphisms are isomorphisms away from the fibers over Σ_0. We observe that

$$\mathcal{P}^{e,\circ}_{\mathscr{U}} \cong \mathfrak{P}^{e,\circ} \times_{\mathbf{BC}^*_\omega \times \mathcal{A}_{\max}} C^\circ_{\mathscr{U}}. \tag{4.16}$$

Consider the line bundle $\mathbf{L}_{\max} := \mathcal{O}_{\mathcal{A}_{\max}}(-\Delta_{\max})$, whose pull-backs will be denoted again by $\mathbf{L}_{\max}$. Form the line bundle $\widetilde{\omega}_{\mathscr{U}} := \omega_{\underline{C^\circ_{\mathscr{U}}}/\underline{\mathscr{U}}} \otimes \mathbf{L}_{\max}^{-\otimes\tilde{r}}$, and consider the sequence of morphisms of line bundles over $C^\circ_{\mathscr{U}}$

$$\widetilde{\omega}_{\mathscr{U}} \longrightarrow \widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty) \longrightarrow \widetilde{\omega}^{\log}_{\mathscr{U}} := \widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty + \Sigma_0) = \omega^{\log}_{C^\circ_{\mathscr{U}}/\mathscr{U}} \otimes \mathbf{L}_{\max}^{-\otimes\tilde{r}} \cong \mathcal{V}|_{C^\circ_{\mathscr{U}}},$$

where $\mathcal{V}$ is the line bundle as in (4.3). We identify these line bundles with their total spaces equipped with the log structures pulled back from $C^\circ_{\mathscr{U}}$.

Lemma 4.3. *There is a canonical commutative diagram of arrows over* $C^\circ_{\mathscr{U}}$

$$\begin{tikzcd} \mathcal{P}^{e,\circ}_{\mathscr{U},reg} \arrow[r, "\widetilde{\mathcal{W}}_-"] \arrow[d] & \widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty) \arrow[d] \\ \mathcal{P}^{e,\circ}_{\mathscr{U}} \arrow[r, "\widetilde{\mathcal{W}}"] & \widetilde{\omega}^{\log}_{\mathscr{U}} \end{tikzcd} \tag{4.17}$$

where the right vertical arrow is given by twisting along markings, and the bottom arrow is obtained by pulling back (4.4).

Proof. This is identical to [18, Lemma 3.13]. □

4.3.2. *A canonical factorization.* Consider the morphism $\mathfrak{b}\colon \mathcal{A}^e \to \mathcal{A} \times \mathcal{A}_{\max} \cong \mathcal{A} \times \mathcal{A}$ as in (4.2). Let $\mathcal{E}_{\mathfrak{b}}$ be the exceptional divisor of $\mathfrak{b}$, and $\widehat{\mathcal{A}} \subset \mathcal{A}^e$ be the proper transform of $\mathcal{A} \times \Delta_{\max}$. Recall that $\infty^e_{\mathcal{A}} \subset \mathcal{A}^e$ is the proper transform of $\infty_{\mathcal{A}} \times \mathcal{A}_{\max}$.

Lemma 4.4. *Let* $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ *be a* ⅄*-punctured map over* S *with non-positive contact orders along markings. Then there is a canonical factorization*

$$\begin{tikzcd} & \mathcal{A}^e \arrow[d] \\ C^\circ \arrow[ur, bend left, "\mathfrak{f}^e"] \arrow[r, "(\mathfrak{f}{,}e_{\max})"'] & \mathcal{A} \times \mathcal{A}_{\max} \end{tikzcd}$$

with the following properties

(1) $\mathfrak{f}^e$ *factors through the open substack* $\mathcal{A}^{e,\circ} := \mathcal{A}^e \setminus \infty^e_{\mathcal{A}}$.

(2) For any geometric point $s \in S$*, an irreducible component* $Z \subset C^\circ_s$ *over* s *dominates* $\mathcal{E}_{\mathfrak{b}}$ *via* $\mathfrak{f}^e$ *iff* Z *is maximally degenerate with a non-zero degeneracy.*

(3) For any $x \in \mathbf{L}_\infty(G)$ *with the corresponding gerbe* $p_x \subset C^\circ$*, we have the image* $\mathfrak{f}^e(p_x) \subset \widehat{\mathcal{A}} \cap \mathcal{E}_{\mathfrak{b}}$.

Proof. Consider the log ideal $\mathcal{K} = \mathcal{M}_{\mathcal{A}\times\mathcal{A}_{\max}} \setminus \mathcal{O}^*$. The morphism $\mathfrak{b}$ is the log blow-up with respect to $\mathcal{K}$. Thus to construct $\mathfrak{f}^e$, it suffices to show that the pull-back log ideal $(\mathfrak{f}^e)^\bullet\mathcal{K}$ is principal, i.e. locally generated by a single element.

Away from the punctured points, the statements (1) and (2) follow by the same argument as in [20, Lemma 3.22]. Now consider a punctured marking $p_x \subset C^\circ$ as in (3). The morphism f^e is well-defined along p_x iff locally around p_x the element $(e_{\max} - \bar{\mathfrak{f}}^\flat(1_{\mathcal{A}})) \in \overline{\mathcal{M}}^{gp}_{C^\circ}$ belongs to $\overline{\mathcal{M}}_{C^\circ}$, where $1_{\mathcal{A}}$ is the generator of $\Gamma(\mathcal{A}, \overline{\mathcal{M}}_{\mathcal{A}}) \cong \mathbb{N}$. We check this locally as follows.

Consider a geometric point $s \in S$ over which $p_{x,s}$ is contained in a component $Z \subset C^\circ_s$ with degeneracy e_Z. Since $e_{\max}$ is the uniform maximal degeneracy, we have $(e_{\max} - e_Z) \in \overline{\mathcal{M}}_S$. Together with the assumption $\mathbf{c}(x) < 0$, we deduce that

$$e_{\max} - \overline{\mathfrak{f}}^{\flat}(1_{\mathcal{A}}) = e_{\max} - (e_Z + \mathbf{c}(x) \cdot \sigma_x) = (e_{\max} - e_Z) - \mathbf{c}(x) \cdot \sigma_x \in \overline{\mathcal{M}}_{C^\circ, \bar{p}_{x,s}} \tag{4.18}$$

where $\bar{p}_{x,s}$ is the strict geometric point above $p_{x,s}$, and $\sigma_x \in \overline{\mathcal{M}}_{C^\circ}$ is the element corresponding to the local coordinate whose vanishing defines $\bar{p}_{x,s}$. Thus by the universal property of log blow-ups, $\mathfrak{f}^e$ is the morphism defined by $(\overline{\mathfrak{f}}^e)^\flat \colon (e_{\max} - 1_{\mathcal{A}}) \mapsto (e_{\max} - \overline{\mathfrak{f}}^{\flat}(1_{\mathcal{A}}))$ on the characteristic level.

Finally, we notice that $\widehat{\mathcal{A}} \cap \mathcal{E}_{\mathfrak{b}}$ is precisely the locus where $e_{\max} \neq 0$ and $(e_{\max} - 1_{\mathcal{A}}) \neq 0$. As $\mathbf{c}(x) < 0$, the puncture p_x is necessarily contained in a degenerate component, hence $e_{\max} \neq 0$ along p_x. Furthermore as the image of $(e_{\max} - 1_{\mathcal{A}})$, the element (4.18) is nowhere zero. This finishes the proof of (3). ☐

Now consider the morphism $\mathfrak{P}^{e,\circ} \to \mathfrak{P} \times \mathcal{A}_{\max}$ as in (4.3) fitting into the Cartesian diagram

$$\begin{tikzcd} \mathfrak{P}^{e,\circ} \arrow[r] \arrow[d] & \mathcal{A}^{e,\circ} \arrow[d, "\mathfrak{b}"] \\ \mathfrak{P} \times \mathcal{A}_{\max} \arrow[r] & \mathcal{A} \times \mathcal{A}_{\max} \end{tikzcd} \tag{4.19}$$

with strict horizontal arrows. Let $\mathfrak{E}^\circ \subset \mathfrak{P}^{e,\circ}$ be the pre-image of $\mathcal{E}_{\mathfrak{b}}$, and $\widehat{\mathfrak{P}} \subset \mathfrak{P}^{e,\circ}$ be the pre-image of $\widehat{\mathcal{A}}$.

Lemma 4.5. *There is a canonical factorization*

$$\begin{tikzcd} & \mathfrak{P}^{e,\circ} \arrow[d] \\ C^\circ_{\mathscr{U}} \arrow[ru, bend left, "f^e_{\mathscr{U}}"] \arrow[r, "(f_{\mathscr{U}}{,}e_{\max})"'] & \mathfrak{P} \times \mathcal{A}_{\max} \end{tikzcd} \tag{4.20}$$

such that

1. *For an irreducible component $Z \subset C^\circ_{\mathscr{U}}|_s$ over a geometric point $s \in \mathscr{U}$, we have $f^e_{\mathscr{U}}(\eta_Z) \subset \mathfrak{E}^\circ \setminus \widehat{\mathfrak{P}}$ for $\eta_Z \in Z$ the generic point iff Z is maximally degenerated with a non-trivial degeneracy.*
2. *For any $x \in \mathbf{L}_\infty(G)$, we have $f_{\mathscr{U}}(p_x) \subset \widehat{\mathfrak{P}} \cap \mathfrak{E}^\circ$.*
3. *For any $x \in \mathbf{L}_0(G)$, we have $f_{\mathscr{U}}(p_x) \subset \mathbf{0}_{\mathfrak{P}} \times \mathcal{A}_{\max}$.*

Proof. The existence of $f^e_{\mathscr{U}}$ and statements (1) and (2) follow from Lemma 4.4 and (4.19). Since $\overline{\gamma}_x$ is a 0-sector for any $x \in \mathbf{L}_0(G)$, we obtain (3). ☐

4.3.3. *The canonical cosection.* By (4.16), the morphism $f^e_{\mathscr{U}}$ is equivalent to a section

$$f_{\mathscr{U},\mathcal{P}^e} \colon C^\circ_{\mathscr{U}} \to \mathcal{P}^{e,\circ}_{\mathscr{U}}$$

of the projection $\mathcal{P}^{e,\circ}_{\mathscr{U}} \to C^\circ_{\mathscr{U}}$. Lemma 4.5 (3) implies a canonical factorization

$$\begin{tikzcd} C^\circ_{\mathscr{U}} \arrow[rr, "f_{\mathscr{U},\mathcal{P}}"] \arrow[rd, "f_{\mathscr{U},-}"'] & & \mathcal{P}^{e,\circ}_{\mathscr{U}} \\ & \mathcal{P}^{e,\circ}_{\mathscr{U},reg} \arrow[ru] & \end{tikzcd} \tag{4.21}$$

Using (4.17), we obtain a morphism of vector bundles over $C^\circ_{\mathscr{U}}$

$$f^*_{\mathscr{U},-}\, \mathrm{d}\widetilde{\mathcal{W}}_- \colon f^*_{\mathscr{U},-} T_{\mathcal{P}^{e,\circ}_{\mathscr{U},reg}/C^\circ_{\mathscr{U}}} \longrightarrow (\widetilde{\mathcal{W}}_- \circ f_{\mathscr{U},-})^* T_{\widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty)/C^\circ_{\mathscr{U}}} \cong \widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty).$$

The following lemma is the main difference comparing to the reduction in the log case of [18] due to the presence of punctures.

Lemma 4.6. *There is a canonical factorization*

$$\begin{tikzcd} f^*_{\mathscr{U},-}T_{\mathcal{P}^{e,\circ}_{\mathscr{U},reg}/C^\circ_{\mathscr{U}}} \arrow[rr, "f^*_{\mathscr{U},-}\mathrm{d}\widetilde{\mathcal{W}}_-"] \arrow[dr, bend right] & & \widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty) \\ & \widetilde{\omega}_{\mathscr{U}} \arrow[ur, bend right] & \end{tikzcd} \tag{4.22}$$

Proof. For any $x \in \mathbf{L}_\infty(G)$, consider the gerbe $p_x \subset C^\circ_{\mathscr{U}}$. By Lemma 4.5 (3) we have the image $f_{\mathscr{U},-}(p_x) \subset \widehat{\mathfrak{P}} \times_{\mathbf{BC}^*_\omega} C^\circ_{\mathscr{U}}$. Since (4.4) contracts $\widehat{\mathfrak{P}}$ to the zero section of $\mathcal{V}$, as the pull-back of (4.4), the morphism $\widetilde{\mathcal{W}}_-$ contacts $\widehat{\mathfrak{P}} \times_{\mathbf{BC}^*_\omega} C^\circ_{\mathscr{U}}$ to the zero section of $\widetilde{\omega}_{\mathscr{U}}(\Sigma_\infty)$. This implies that the morphism $f^*_{\mathscr{U},-}\mathrm{d}\widetilde{\mathcal{W}}_-$ degenerates along p_x, hence the factorization (4.22). □

Lemma 4.7. *There is a canonical exact sequence*

$$0 \to f^*_{\mathscr{U}}T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0) \to f^*_{\mathscr{U},-}T_{\mathcal{P}^{e,\circ}_{\mathscr{U},reg}/C^\circ_{\mathscr{U}}} \to T_{\mathfrak{X}/\mathbf{BC}^*_\omega}|_{\Sigma_0} \to 0.$$

Proof. This is similar to [18, Lemma 3.14] by applying (4.16). We omit the details. □

Combining Lemma 4.6 and Lemma 4.7, we obtain a composition

$$f^*_{\mathscr{U}}T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0) \to f^*_{\mathscr{U},-}T_{\mathcal{P}^{e,\circ}_{\mathscr{U},reg}/C^\circ_{\mathscr{U}}} \to \widetilde{\omega}_{\mathscr{U}}.$$

Applying $R\pi_{\mathscr{U},*}$ and the projection formula, we obtain

$$\sigma^\bullet_{\mathscr{U}/\mathfrak{U}} \colon \mathbb{E}_{\mathscr{U}/\mathfrak{U}} = R\pi_{\mathscr{U},*}f^*_{\mathscr{U}'}T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0) \longrightarrow R\pi_{\mathscr{U},*}\widetilde{\omega}_{\mathscr{U}} \cong R\pi_{\mathscr{U},*}(\omega_{\underline{C^\circ_{\mathscr{U}}}/\underline{\mathscr{U}}}) \otimes \mathbf{L}^{-\otimes\tilde{r}}_{\max}$$

Applying Serre duality, we obtain

$$R^1\pi_{\mathscr{U},*}(\omega_{\underline{C^\circ_{\mathscr{U}}}/\underline{\mathscr{U}}}) \otimes \mathbf{L}^{-\otimes\tilde{r}}_{\max} \cong \bigoplus_{G_i \subset G} \mathbf{L}^{-\otimes\tilde{r}}_{\max}$$

where the direct sum runs through connected components $G_i \subset G$. Taking H^1 of $\sigma^\bullet_{\mathscr{U}/\mathfrak{U}}$, we obtain the *canonical cosection*

$$\sigma_{\mathscr{U}/\mathfrak{U}} \colon \mathrm{Obs}_{\mathscr{U}/\mathfrak{U}} := H^1(\mathbb{E}_{\mathscr{U}/\mathfrak{U}}) \longrightarrow \bigoplus_{G_i \subset G} \mathbf{L}^{\otimes-\tilde{r}}_{\max} \longrightarrow \mathbf{L}^{-\otimes\tilde{r}}_{\max}, \tag{4.23}$$

where the arrow on the right is the sum of identity $\mathbf{L}^{\otimes-\tilde{r}}_{\max} \to \mathbf{L}^{\otimes-\tilde{r}}_{\max}$.

Proposition 4.8. *Suppose W has proper critical locus. Then the degeneracy locus of $\sigma_{\mathscr{U}/\mathfrak{U}}$ is contained in $\mathscr{U} \setminus (\mathscr{U} \times_{\mathcal{A}_{\max}} \Delta_{\max})$.*

Proof. In case of log R-maps, this is precisely [18, Prop. 3.18]. We now verify the statement for general punctured R-maps. As the target of $\sigma_{\mathscr{U}/\mathfrak{U}}$ is a line bundle, it suffices to check the surjectivity of $\sigma_{\mathscr{U}/\mathfrak{U}}$ at any geometric point $s \in \mathscr{U} \times_{\mathcal{A}_{\max}} \Delta_{\max}$. Denote by $f := \sqcup f_i \colon C^\circ = \sqcup_i C^\circ_i \to \mathfrak{P}$ the punctured R-map over s where $f_i \colon C^\circ_i \to \mathfrak{P}$ corresponds to the connected component $G_i \subset G$. At least one of the connected component of C°, say C°_k contains a maximally degenerate component. By the construction of (4.23), it suffices to show that

$$H^1(f^*_kT_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0)) \longrightarrow \mathbf{L}^{-\otimes\tilde{r}}_{\max}$$

is surjective. The rest of the proof is identical to [18, Prop. 3.18], and is omitted. □

4.4. **Reduction by the canonical cosection.** We next construct the reduced theory under the following assumption:

Assumption 4.9. $\boldsymbol{\tau}$ has at least one degenerate vertex.

Let $\sigma_{\boldsymbol{\tau}}$ be the basic cone of $\boldsymbol{\tau}$. Then the above assumption is equivalent to that the maximal degeneracy $e_{\max} \in \sigma_{\boldsymbol{\tau}}$ is non-zero. By Definition (3.11) (3), this is further equivalent to a factorization

(4.24)
$$\begin{tikzcd}
\mathfrak{U} \arrow[rr] \arrow[dr] & & \mathcal{A}_{\max} \\
& \Delta_{\max} \arrow[ur] &
\end{tikzcd}$$

where the top horizontal arrow is defined by (3.23).

Define the boundary complex $\mathbb{F} := \mathbf{L}_{\max}^{-\otimes \tilde{r}}[-1]$. The composition

(4.25)
$$\mathbb{E}_{\mathscr{U}/\mathfrak{U}} \longrightarrow H^1(\mathbb{E}_{\mathscr{U}/\mathfrak{U}})[-1] \xrightarrow{\sigma_{\mathscr{U}/\mathfrak{U}}[-1]} \mathbb{F}_{\mathscr{U}} := \mathbb{F}|_{\mathscr{U}}$$

leads to a distinguished triangle

(4.26)
$$\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}} \longrightarrow \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \longrightarrow \mathbb{F}_{\mathscr{U}} \xrightarrow{[1]}$$

defining the object $\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}$. Our next goal is to establish the following result.

Theorem 4.10. *Let $\boldsymbol{\tau}$ be a decorated type of compact type as in (4.6). Further assume that $\boldsymbol{\tau}$ satisfies Assumption 4.9. Then there is a canonical factorization*

(4.27)
$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}/\mathfrak{U}} \arrow[rr, "\varphi_{\boldsymbol{\tau}}"] \arrow[dr, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}"'] & & \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \\
& \mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}} \arrow[ur] &
\end{tikzcd}$$

such that $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$ defines a perfect obstruction theory of $\mathscr{U} \to \mathfrak{U}$, called the reduced perfect obstruction theory*.*

After some preparation, we will prove this theorem in §4.4.2.

Remark 4.11. The reduced perfect obstruction theory $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$ is different than the reduced theory in [18, §3.6.3] in that the latter does not assume the factorization (4.24) or equivalently Assumption (4.9). On the level of virtual cycles, this factorization results in a formula (4.40) which is different than [18, Theorem 3.21].

On the other hand, when $\boldsymbol{\tau}$ is a tropical type of log R-maps, or equivalently there is no negative contact orders in $\boldsymbol{\tau}$, the construction of $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$ is identical to the reduced perfect obstruction theory of the boundary in [18, §3.8]. This fact will be used in proving Theorem 4.15.

4.4.1. *The twisted hodge bundle.* Consider the stack $\mathfrak{U}$ as in (4.8) and its universal family

$$\mathfrak{f}_{\mathfrak{U}} \colon C^{\circ}_{\mathfrak{U}} \longrightarrow \mathcal{A}, \qquad \pi_{\mathfrak{U}} \colon C^{\circ}_{\mathfrak{U}} \longrightarrow \mathfrak{U}.$$

Define the line bundle

$$\widetilde{\omega}_{\mathfrak{U}} := \omega_{\underline{C^{\circ}_{\mathfrak{U}}}/\underline{\mathfrak{U}}} \otimes \mathbf{L}_{\max}^{\otimes -\tilde{r}}$$

over $C^{\circ}_{\mathfrak{U}}$. We may view $\widetilde{\omega}_{\mathfrak{U}}$ as a log stack with its log structure pulled back from $C^{\circ}_{\mathfrak{U}}$. Applying the projection formula, we obtain the *twisted hodge bundle*

$$\mathfrak{H} := R^0\pi_{\mathfrak{U},*}\widetilde{\omega}_{\mathfrak{U}} = R^0\pi_{\mathfrak{U},*}\left(\omega_{\underline{C^{\circ}_{\mathfrak{U}}}/\underline{\mathfrak{U}}}\right) \otimes \mathbf{L}_{\max}^{-\otimes \tilde{r}}.$$

over $\mathfrak{U}$. Viewing $\mathfrak{H}$ as a log stack with the log structure pulled back from $\mathfrak{U}$, it parameterizes sections of $\widetilde{\omega}_{\mathfrak{U}} \to C^{\circ}_{\mathfrak{U}}$. Consider the commutative diagram

$$\begin{tikzcd}
& \omega_{\underline{C^{\circ}_{\mathfrak{H}}}/\underline{\mathfrak{H}}} \otimes \mathbf{L}_{\max}^{\otimes -\tilde{r}} \arrow[r] \arrow[d] & \mathcal{L}_{\omega} \boxtimes \mathbf{L}_{\max}^{\otimes -\tilde{r}} \arrow[d] \\
C^{\circ}_{\mathfrak{H}} \arrow[ur, bend left, "\mathbf{s}_{\mathfrak{H}}"] \arrow[r, equal] & C^{\circ}_{\mathfrak{H}} \arrow[r] & \mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}
\end{tikzcd} \tag{4.28}$$

where $\pi_{\mathfrak{H}} \colon C^{\circ}_{\mathfrak{H}} := C^{\circ}_{\mathfrak{U}} \times_{\mathfrak{U}} \mathfrak{H} \to \mathfrak{H}$ is the universal punctured curve, and $\rho_{\mathfrak{H}}$ is the universal section over $\mathfrak{H}$.

On the other hand, composing (4.4) and (4.20), we obtain a commutative square

$$\begin{tikzcd}
& \mathcal{L}_{\omega} \boxtimes \mathbf{L}_{\max}^{\otimes -\tilde{r}} \arrow[d] \\
C^{\circ}_{\mathscr{U}} \arrow[ur, bend left, "\widetilde{W} \circ f^{e}_{\mathscr{U}}"] \arrow[r] & \mathbf{BC}^{*}_{\omega} \times \mathcal{A}_{\max}
\end{tikzcd}$$

over $C^{\circ}_{\mathscr{U}}$. Recall that $\widetilde{W}$ as in (4.4) constracts $\widetilde{\mathfrak{P}}$. By Lemma 4.5 (2), (3), there is a factorization

$$\begin{tikzcd}
C^{\circ}_{\mathscr{U}} \arrow[rr, "\widetilde{W} \circ f^{e}_{\mathscr{U}}"] \arrow[dr, bend right, "\mathbf{s}_{\mathscr{U}}"'] & & \mathcal{L}_{\omega} \boxtimes \mathbf{L}_{\max}^{\otimes -\tilde{r}} \\
& \omega_{\underline{C^{\circ}_{\mathscr{U}}}/\underline{\mathscr{U}}} \otimes \mathbf{L}_{\max}^{\otimes -\tilde{r}} \arrow[ur, bend right] &
\end{tikzcd}$$

This defines a tautological morphism $\mathscr{U} \to \mathfrak{H}$ along which $\mathbf{s}_{\mathfrak{H}}$ pulls back to $\mathbf{s}_{\mathscr{U}}$.

By [13, Prop. 2.5], the morphism $\mathfrak{H} \to \mathfrak{U}$ admits a perfect obstruction theory

$$\varphi_{\mathfrak{H}/\mathfrak{U}} \colon \mathbb{T}_{\mathfrak{H}/\mathfrak{U}} \longrightarrow \mathbb{E}_{\mathfrak{H}/\mathfrak{U}} := \pi_{\mathfrak{H},*}(\widetilde{\omega}_{\mathfrak{U}}|_{C^{\circ}_{\mathfrak{H}}}).$$

Note that $\mathbb{T}_{\mathfrak{H}/\mathfrak{U}} \cong T_{\mathfrak{H}/\mathfrak{U}} \cong \mathfrak{H}|_{\mathfrak{H}}$ where the last term is the corresponding vector bundle over $\mathfrak{H}$. Consider the composition

$$\mathbb{E}_{\mathfrak{H}/\mathfrak{U}} \longrightarrow H^{1}(\mathbb{E}_{\mathfrak{H}/\mathfrak{U}})[-1] \longrightarrow \mathbb{F}_{\mathfrak{H}} := \mathbb{F}|_{\mathfrak{H}} \tag{4.29}$$

where the second arrow is obtained by shifting the following surjection

$$\sigma_{\mathfrak{H}/\mathfrak{U}} \colon H^{1}(\mathbb{E}_{\mathfrak{H}/\mathfrak{U}}) = R^{1}\pi_{\mathfrak{H},*}\left(\widetilde{\omega}_{\mathfrak{U}}|_{C^{\circ}_{\mathfrak{H}}}\right) = \bigoplus_{G_i \subset G} \mathbf{L}_{\max}^{\otimes -\tilde{r}}|_{\mathfrak{H}} \to \mathbf{L}_{\max}^{\otimes -\tilde{r}}|_{\mathfrak{H}}$$

with the last arrow defined as in (4.23).

Lemma 4.12. *The following composition is the zero morphism*

$$\mathbb{T}_{\mathfrak{H}/\mathfrak{U}} \longrightarrow \mathbb{E}_{\mathfrak{H}/\mathfrak{U}} \longrightarrow \mathbb{F}_{\mathfrak{H}} \tag{4.30}$$

Proof. Since $\mathfrak{H} \to \mathfrak{U}$ is a vector bundle, we have $\mathbb{T}_{\mathfrak{H}/\mathfrak{U}} \cong H^{0}(\mathbb{E}_{\mathfrak{H}/\mathfrak{U}})$. Thus, the result follows from the distinguished triangle

$$H^{0}(\mathbb{E}_{\mathfrak{H}/\mathfrak{U}}) \longrightarrow \mathbb{E}_{\mathfrak{H}/\mathfrak{U}} \longrightarrow H^{1}(\mathbb{E}_{\mathfrak{H}/\mathfrak{U}})[-1] \xrightarrow{[1]} .$$

□

Lemma 4.13. *There is a canonical commutative diagram*

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}/\mathfrak{U}} \arrow[r, "\varphi_{\boldsymbol{\tau}}"] \arrow[d] & \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \arrow[r, "(4.25)"] \arrow[d] & \mathbb{F}_{\mathscr{U}} \arrow[d, "\cong"] \\
\mathbb{T}_{\mathfrak{H}/\mathfrak{U}}|_{\mathscr{U}} \arrow[r, "\varphi_{\mathfrak{H}/\mathfrak{U}}|_{\mathscr{U}}"'] & \mathbb{E}_{\mathfrak{H}/\mathfrak{U}}|_{\mathscr{U}} \arrow[r, "(4.29)"'] & \mathbb{F}_{\mathfrak{H}}|_{\mathscr{U}}
\end{tikzcd} \tag{4.31}$$

where the vertical arrow on the left is induced by $\mathscr{U} \to \mathfrak{H}$*,*

Proof. Applying the construction of $\mathcal{P}^{e,\circ}_{\mathscr{U},reg}$ in (4.15) with $\mathscr{U}$ replaced by $\mathfrak{U}$, we obtain a stack $\mathcal{P}^{e,\circ}_{\mathfrak{U},reg}$ over $C^\circ_{\mathfrak{U}}$ such that $\mathcal{P}^{e,\circ}_{\mathscr{U},reg} = \mathcal{P}^{e,\circ}_{\mathfrak{U},reg} \times_{C^\circ_{\mathfrak{U}}} C^\circ_{\mathscr{U}}$. Similar to (4.17), we may construct a morphism $\widetilde{W}_{\mathfrak{U},-} \colon \mathcal{P}^{e,\circ}_{\mathfrak{U},reg} \to \widetilde{\omega}_{\mathfrak{U}}(\Sigma_\infty)$ pulling back to $\widetilde{W}_-$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{U} \arrow[d] & C^\circ_{\mathscr{U}} \arrow[l] \arrow[r, "f_{\mathscr{U},-}"] \arrow[d] & \mathcal{P}^{e,\circ}_{\mathfrak{U},reg} \arrow[d, "\widetilde{W}_{\mathfrak{U},-}"] \\
\mathfrak{H} \arrow[d] & C^\circ_{\mathfrak{H}} \arrow[l] \arrow[r] \arrow[d] & \widetilde{\omega}_{\mathfrak{U}}(\Sigma_\infty) \arrow[d] \\
\mathfrak{U} & C^\circ_{\mathfrak{U}} \arrow[l] \arrow[r, equal] & C^\circ_{\mathfrak{U}}
\end{tikzcd} \tag{4.32}$$

where by abuse of notations $f_{\mathscr{U},-}$ denotes the arrow induced by (4.21), and the two left squares are Cartesian with strict vertical arrows.

Consider the following commutative diagram

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}/\mathfrak{U}}|_{C^\circ_{\mathscr{U}}} \cong \mathbb{T}_{C^\circ_{\mathscr{U}}/C^\circ_{\mathfrak{U}}} \arrow[r] \arrow[d] \arrow[rr, bend left, "\mathrm{d}\, f_{\mathscr{U},-}"] & f^*_{\mathscr{U}} T_{\mathfrak{P}/\mathbf{BC}^*_\omega}(-\Sigma_0) \arrow[r] \arrow[d] & f^*_{\mathscr{U},-} T_{\mathcal{P}^{e,\circ}_{\mathfrak{U},reg}/C^\circ_{\mathfrak{U}}} \arrow[d, "\mathrm{d}\, \widetilde{W}_{\mathfrak{U},-}"] \\
\mathbb{T}_{\mathfrak{H}/\mathfrak{U}}|_{C^\circ_{\mathscr{U}}} \cong \mathbb{T}_{C^\circ_{\mathfrak{H}}/C^\circ_{\mathfrak{U}}} \arrow[r, "\mathrm{d}\, \mathbf{s}_{\mathfrak{H}}"] \arrow[rr, bend right] & \omega_{\underline{C^\circ_{\mathscr{U}}}/\underline{\mathscr{U}}} \otimes \mathbf{L}_{\max}^{\otimes -\tilde{r}} \arrow[r] & \omega_{\underline{C^\circ_{\mathscr{U}}}/\underline{\mathscr{U}}}(\Sigma_\infty) \otimes \mathbf{L}_{\max}^{\otimes -\tilde{r}}
\end{tikzcd} \tag{4.33}$$

where the outer square is obtained from (4.32) by taking differentiation, the factorization on the top is from Lemma 4.7, and the factorization on the bottom is from (4.28). The square on the right hand side of (4.31) follows from applying $\pi_{\mathscr{U},*}$ to the square on the right.

Finally, the square on the left hand side of (4.31) follows from $\mathbf{s}_{\mathscr{U}} = \mathbf{s}_{\mathfrak{H}}|_{C^\circ_{\mathscr{U}}}$. ☐

4.4.2. *Proof of Theorem 4.10.* Lemma 4.13 provides a commutative diagram of solid arrows

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}/\mathfrak{U}} \arrow[r, dashed, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}"'] \arrow[rr, bend left, "\varphi_{\boldsymbol{\tau}}"] \arrow[dr, bend right] & \mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}} \arrow[r] \arrow[d] & \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \arrow[r] \arrow[d] & \mathbb{F}_{\mathscr{U}} \arrow[r, "{[1]}"] \arrow[d, equal] & {} \\
 & \mathbb{T}_{\mathfrak{H}/\mathfrak{U}}|_{\mathscr{U}} \arrow[r] & \mathbb{E}_{\mathfrak{H}/\mathfrak{U}}|_{\mathscr{U}} \arrow[r] & \mathbb{F}_{\mathfrak{H}}|_{\mathscr{U}} &
\end{tikzcd} \tag{4.34}$$

The commutativity implies that the composition $\mathbb{T}_{\mathscr{U}/\mathfrak{U}} \to \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \to \mathbb{F}_{\mathscr{U}}$ factors through the zero morphism (4.30). Hence we obtain the dashed arrow $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$ making the diagram commutative. Next, we verify that $\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}$ is perfect in $[0,1]$.

Since $\mathbb{E}_{\mathscr{U}/\mathfrak{U}}$ is perfect in $[0,1]$ and $\mathbb{F}_{\mathscr{U}}$ is a vector bundle shifted by $[-1]$, $\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}$ is perfect in $[0,2]$. It suffices to show that $H^2(\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}) = 0$. Since $\sigma_{\mathscr{U}/\mathfrak{U}}$ is surjective by Proposition 4.8, this follows from the long exact sequence

$$H^1(\mathbb{E}_{\mathscr{U}/\mathfrak{U}}) \twoheadrightarrow H^1(\mathbb{F}_{\mathscr{U}}) \longrightarrow H^2(\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}) \longrightarrow H^2(\mathbb{E}_{\mathscr{U}/\mathfrak{U}}) = 0.$$

To verify that $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$ is a perfect obstruction theory, we first take the H^0 of (4.34), and obtain

$$\begin{tikzcd}
 & & H^0(\mathbb{T}_{\mathscr{U}/\mathfrak{U}}) \arrow[dl, "H^0(\varphi^{\mathrm{red}}_{\boldsymbol{\tau}})"'] \arrow[d, "H^0(\varphi_{\boldsymbol{\tau}})"] & \\
0 \arrow[r] & H^0(\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}) \arrow[r, "\cong"'] & H^0(\mathbb{E}_{\mathscr{U}/\mathfrak{U}}) \arrow[r] & H^0(\mathbb{F}_{\mathscr{U}}) = 0.
\end{tikzcd}$$

Since $H^0(\varphi_{\boldsymbol{\tau}})$ is an isomorphism, $H^0(\varphi^{\mathrm{red}}_{\boldsymbol{\tau}})$ is also an isomorphism.

Finally, we take the H^1 of (4.34), and obtain

$$\begin{tikzcd}
 & & H^1(\mathbb{T}_{\mathscr{U}/\mathfrak{U}}) \arrow[dl, "H^1(\varphi^{\mathrm{red}}_{\boldsymbol{\tau}})"'] \arrow[d, "H^1(\varphi_{\boldsymbol{\tau}})"] & & \\
0 \arrow[r] & H^1(\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}}) \arrow[r] & H^1(\mathbb{E}_{\mathscr{U}/\mathfrak{U}}) \arrow[r] & H^1(\mathbb{F}_{\mathscr{U}}) = \mathbb{F}_{\mathscr{U}}[1] \arrow[r] & 0
\end{tikzcd}$$

The injectivity of $H^1(\varphi_{\boldsymbol{\tau}})$ implies the injectivity of $H^1(\varphi^{\mathrm{red}}_{\boldsymbol{\tau}})$. Therefore, $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$ is a perfect obstruction theory of $\mathscr{U} \to \mathfrak{U}$. ☐

4.5. **A compatible reduction.** Consider the composition

$$\mathbb{E}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} \to \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \to \mathbb{F}_{\mathscr{U}}$$

where the first arrow is given by (2.56) and (4.11). This leads to a distinguished triangle

$$\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} \longrightarrow \mathbb{E}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} \longrightarrow \mathbb{F}_{\mathscr{U}} \xrightarrow{[1]},$$

defining $\mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}}$. This triangle fits in a commutative diagram of solid arrows

(4.35)
$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} \arrow[rr] \arrow[dr, dashed, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau},\mathrm{ev}}"] \arrow[ddr, bend right, "\varphi_{\boldsymbol{\tau},\mathrm{ev}}"'] & & \mathbb{T}_{\mathscr{U}/\mathfrak{U}} \arrow[rr] \arrow[dr, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}"] \arrow[ddr, bend right, "\varphi_{\boldsymbol{\tau}}"'] & & T_{\mathfrak{U}^{\mathrm{ev}}/\mathfrak{U}} \arrow[r, "{[1]}"] \arrow[dr, equal] & {} & \\
 & \mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} \arrow[rr] \arrow[d] & & \mathbb{E}^{\mathrm{red}}_{\mathscr{U}/\mathfrak{U}} \arrow[rr] \arrow[d] & & T_{\mathfrak{U}^{\mathrm{ev}}/\mathfrak{U}} \arrow[r] \arrow[d, equal] & {} \\
 & \mathbb{E}_{\mathscr{U}/\mathfrak{U}^{\mathrm{ev}}} \arrow[rr] \arrow[d] & & \mathbb{E}_{\mathscr{U}/\mathfrak{U}} \arrow[rr] \arrow[d] & & T_{\mathfrak{U}^{\mathrm{ev}}/\mathfrak{U}} \arrow[r] & {} \\
 & \mathbb{F}_{\mathscr{U}} \arrow[rr, equal] & & \mathbb{F}_{\mathscr{U}} & & &
\end{tikzcd}$$

where the top three rows are distinguished triangles, the morphisms between the first and third rows form a morphism of distinguished triangles by pulling back (2.56). Thus, we obtain the dashed arrow $\varphi^{\mathrm{red}}_{\boldsymbol{\tau},\mathrm{ev}}$ making the diagram commutative, so that the arrows between the first and the second row form a morphism of distinguished triangles. In particular, $\varphi^{\mathrm{red}}_{\boldsymbol{\tau},\mathrm{ev}}$ is a perfect obstruction theory of $\mathscr{U} \to \mathfrak{U}^{\mathrm{ev}}$ compatible with $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$. We will also refer to $\varphi^{\mathrm{red}}_{\boldsymbol{\tau},\mathrm{ev}}$ as the *reduced perfect obstruction theory.* It will be used in the gluing construction in §5.

4.6. **The reduced theory of ⅄-types.** Let $\boldsymbol{\tau}_{⅄} = (\boldsymbol{\tau}, \mathbf{V}_{\max}(G))$ be a decorated ⅄-type where $\boldsymbol{\tau}$ is given as in §4.2. Define the log stacks $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{⅄})$, $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_{⅄})$ and $\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{⅄})$

via the diagram

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge}) \arrow[r] \arrow[d] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_{\curlywedge}) \arrow[r] \arrow[d] & \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge}) \arrow[r] \arrow[d] & \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \arrow[d, "(3.14)"] \\
\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}) \arrow[r] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau) \arrow[r] & \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau), \arrow[r] & \mathfrak{U}(\mathcal{A}, \tau)
\end{tikzcd} \tag{4.36}$$

with Cartesian squares and strict vertical arrows. Pulling back (4.27), we obtain a commutative triangle over $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})$

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge})} \arrow[rr, "\varphi_{\boldsymbol{\tau}_{\curlywedge}}"] \arrow[dr, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}}"'] & & \mathbb{E}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge})} \\
& \mathbb{E}^{\mathrm{red}}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge})} \arrow[ur] &
\end{tikzcd} \tag{4.37}$$

where as the pull-backs of $\varphi_{\boldsymbol{\tau}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}}$, the morphisms $\varphi_{\boldsymbol{\tau}_{\curlywedge}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}}$ define the *canonical* and *reduced* perfect obstruction theories of $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge}) \to \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge})$, respectively.

Further pulling back (4.35), we obtain a commutative triangle

$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_{\curlywedge})} \arrow[rr, "\varphi_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}"] \arrow[dr, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}"'] & & \mathbb{E}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_{\curlywedge})} \\
& \mathbb{E}^{\mathrm{red}}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_{\curlywedge})} \arrow[ur] &
\end{tikzcd} \tag{4.38}$$

where as the pull-backs of $\varphi_{\boldsymbol{\tau}, \mathrm{ev}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}, \mathrm{ev}}$, $\varphi_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}$ define the *canonical* and *reduced* perfect obstruction theories of $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge}) \to \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_{\curlywedge})$ respectively.

By the compatibility of perfect obstruction theories in (4.35), we observe that the canonical perfect obstruction theories $\varphi_{\boldsymbol{\tau}_{\curlywedge}}$ and $\varphi_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}$ are compatible, and the reduced theories $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}$ are compatible as well.

Suppose that $\tau_{\curlywedge}$ is realizable, and hence $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ is equidimensional by Proposition 3.18. In this case, we obtain the canonical and the reduced virtual cycles

$$[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{vir}}, \qquad [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}}, \tag{4.39}$$

defined by $\varphi_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\curlywedge}, \mathrm{ev}}$, respectively.

Corollary 4.14. *Suppose that we are in the situation of Theorem 4.10, and that* $\tau_{\curlywedge}$ *is realizable. Then the two virtual cycles* (4.39) *are related via*

$$[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{vir}} = \tilde{r}\Delta_{\max} \cap [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}}. \tag{4.40}$$

Proof. By (4.35) and its pull-back (4.38), the zero map

$$\mathbb{T}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})/\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})} \longrightarrow \mathbb{F}_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})} := \mathbb{F}|_{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})}$$

is a perfect obstruction theory of the identity of $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})$. Recall that $\mathbb{F} := \mathbf{L}_{\max}^{-\otimes \tilde{r}}[-1]$. The formula follows from [30, Cor. 4.9]. ☐

4.7. Splitting of the reduced boundary virtual cycles.

We consider a decorated type of log R-maps

$$\boldsymbol{\tau}^{\star} = (\tau^{\star}, \overline{\gamma}, \boldsymbol{\beta}) \tag{4.41}$$

where $\tau^{\star}$ is as in (3.24). Further assume that $\boldsymbol{\tau}^{\star}$ is of compact type as in §2.3.1, i.e. for any $x \in \mathbf{L}(G^{\star})$, we have $\mathbf{c}(x) = 0$ and $\overline{\gamma}_x$ is a 0-sector.

Consider a contraction $\tau_{\curlywedge} \vdash \tau^{\star}$ of a $\curlywedge$-type of degenerate rigid tropical curves as in Proposition 3.22. We form a diagram of Cartesian squares with strict arrows

$$\begin{tikzcd}
\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge}) \arrow[r] \arrow[d] & \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge}) \arrow[r] \arrow[d] & \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \arrow[d] \\
\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}) \arrow[r] \arrow[d, "i_{\Delta^{\curlywedge}}"'] & \Delta^{\curlywedge, \overline{\gamma}_{\mathbf{L}_0}}(\tau^{\star}) \arrow[r] \arrow[d] & \Delta^{\curlywedge}(\tau^{\star}) \arrow[d] \\
\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}^{\star}) \arrow[r, "\mathbf{ev}_{\mathbf{L}_0}"] & \mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau^{\star}) \arrow[r] & \mathfrak{U}(\mathcal{A}, \tau^{\star})
\end{tikzcd}$$

where $\Delta^{\curlywedge}(\tau^{\star}) \to \mathfrak{U}(\mathcal{A}, \tau^{\star})$ is given by (3.25), and $\mathfrak{U}(\mathcal{A}, \tau_{\curlywedge}) \to \Delta^{\curlywedge}(\tau^{\star})$ is given by $i_{\tau_{\curlywedge}}$ in Lemma 3.21, and $\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge}) \to \mathfrak{U}(\mathcal{A}, \tau_{\curlywedge})$ is as in (4.7), and the stacks $\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge})$, $\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})$ and $\Delta^{\curlywedge, \overline{\gamma}_{\mathbf{L}_0}}(\tau^{\star})$ are defined as the fs fiber products fitting into this diagram. We further note that $\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})$ agrees with the boundary stack $\Delta_{\mathscr{U}}$ in the notation of [18].

The stack $\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge})$ admits an open and closed decomposition

$$\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge}) = \bigsqcup_{\boldsymbol{\tau}_{\curlywedge}} \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge}) \tag{4.42}$$

where $\boldsymbol{\tau}_{\curlywedge}$ runs through all possible decorated $\curlywedge$-types over $\tau_{\curlywedge}$ admitting contraction to $\boldsymbol{\tau}^{\star}$.

Note that both $\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau^{\star})$ and $\mathfrak{U}^{\overline{\gamma}_{\mathbf{L}_0}}(\mathcal{A}, \tau_{\curlywedge})$ are pure-dimensional. Thus the reduced perfect obstruction theory of Theorem 4.10 defines a reduced virtual cycle $[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}}$ for each $\boldsymbol{\tau}_{\curlywedge}$.

On the other hand, $\Delta^{\curlywedge, \overline{\gamma}_{\mathbf{L}_0}}(\tau^{\star})$ is pure-dimensional. By [18, §3.6.3], $\mathbf{ev}_{\mathbf{L}_0}$ admits a reduced perfect obstruction theory, hence a reduced virtual cycle $[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})]^{\mathrm{red}}$, which decomposes according to tropical types as follows.

Theorem 4.15 (Tropical Splitting formula)**.**

$$i_{\Delta^{\curlywedge},*}[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})]^{\mathrm{red}} = \sum_{\boldsymbol{\tau}_{\curlywedge} \vdash \boldsymbol{\tau}^{\star}} \frac{\mathrm{lcm}_{x \in \mathbf{E}(G)}(\mathbf{c}(x))}{|\operatorname{Aut}(\tau_{\curlywedge})|} i_{\boldsymbol{\tau}_{\curlywedge},*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}}$$

where the summation runs through decorated $\curlywedge$-types $\boldsymbol{\tau}_{\curlywedge} = (\tau_{\curlywedge}, \overline{\gamma}, \boldsymbol{\beta})$ of degenerate rigid tropical curves admitting contractions to $\boldsymbol{\tau}^{\star}$, G is the graph correspondingn to $\tau_{\curlywedge}$, and $i_{\boldsymbol{\tau}_{\curlywedge}} \colon \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge}) \to \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}^{\star})$ is the strict tautological morphism.

Proof. The virtual cycle $[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})]^{\mathrm{red}}$ is defined via the reduced perfect obstruction theory

$$\varphi'_{\Delta^{\curlywedge}} \colon \mathbb{T}_{\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})/\Delta^{\curlywedge, \overline{\gamma}_{\mathbf{L}_0}}(\tau^{\star})} \longrightarrow \mathbb{E}'_{\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})/\Delta^{\curlywedge, \overline{\gamma}_{\mathbf{L}_0}}(\tau^{\star})},$$

constructed in [18, §3.6.3]. It is straightforward to see that the restriction $\varphi'_{\Delta^{\curlywedge}}|_{\mathscr{U}(\boldsymbol{\tau}_{\curlywedge})}$ is the reduced perfect obstruction theory constructed in (4.27), see Remark 4.11.

Denote by $[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge})]^{\mathrm{red}}$ the reduced virtual cycle obtained by pulling back the reduced perfect obstruction theory $\varphi'_{\Delta^{\curlywedge}}$. Applying the virtual push-forward [30], we compute

$$\begin{aligned} i_{\Delta^{\curlywedge},*}[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star})]^{\mathrm{red}} &= \sum_{\tau_{\curlywedge}} \frac{\mathrm{lcm}_{x \in \mathbf{E}(G)}(\mathbf{c}(x))}{|\operatorname{Aut}(\tau_{\curlywedge})|} i_{\tau_{\curlywedge},*}[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge})]^{\mathrm{red}} \\ &= \sum_{\boldsymbol{\tau}_{\curlywedge}} \frac{\mathrm{lcm}_{x \in \mathbf{E}(G)}(\mathbf{c}(x))}{|\operatorname{Aut}(\tau_{\curlywedge})|} i_{\boldsymbol{\tau}_{\curlywedge},*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}} \end{aligned}$$

where $i_{\tau_{\curlywedge}} \colon \Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^{\star}, \tau_{\curlywedge}) \to \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}^{\star})$ is the natural embedding. The first equality follows from Proposition 3.22, and the second one follows from (4.42). This finishes the proof. □

5. DECOMPOSING VIRTUAL CYCLES OF RIGID TROPICAL CURVES

5.1. The set-up and the tropical decomposition formula. Fix a decorated $\Yup$-type

$$\boldsymbol{\tau}_{\Yup} = (\tau_{\Yup}, \overline{\gamma}, \boldsymbol{\beta}), \text{ with } \tau_{\Yup} = (\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}), \boldsymbol{\sigma}, \mathbf{c}, \mathbf{V}_{\max}(G))\,. \tag{5.1}$$

of (possibly non-degenerate) rigid tropical curves, see §3.1.3. We further assume that $\boldsymbol{\tau}_{\Yup}$ is of compact type as in §2.3.1. Denote by $\boldsymbol{\tau}$ (resp. τ) the decorated type (resp. type) given by $\boldsymbol{\tau}_{\Yup}$ (resp. $\tau_{\Yup}$).

The moduli $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\Yup})$ of stable punctured R-maps marked by $\boldsymbol{\tau}_{\Yup}$ admits a canonical $[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\Yup})]^{\mathrm{vir}}$ and a reduced virtual cycle $[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\Yup})]^{\mathrm{red}}$ as in (4.39).

Denote by $n = |\mathbf{L}(G)|$, $\beta = \sum_{V \in \mathbf{V}(G)} \boldsymbol{\beta}(V)$, and $g = \mathbf{g}(\mathbf{G})$ as in (B.8). Let $\mathscr{M}_{g,n}(\mathcal{X}, \beta)$ be the moduli of stable maps to $\mathcal{X}$ from genus g (connected) domain curves with n marked points and curve class β. Thus we obtain a tautological proper morphism

$$F_{\mathscr{M}} \colon \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\Yup}) \longrightarrow \mathscr{M}_{g,n}(\mathcal{X}, \beta), \tag{5.2}$$

by composing R-maps with $\mathfrak{P} \to \mathcal{X}$ and stabilizing. Our goal is to compute $F_{\mathscr{M},*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\Yup})]^{\mathrm{red}}$ in terms of vertex contributions.

5.1.1. *Rigid tropical curves as bipartite graphs.* By (3.5) and Proposition 3.5, the graph G has a bipartite structure in the sense that

$$\mathbf{V}(G) = \mathbf{V}_0(G) \sqcup \mathbf{V}_\infty(G) \tag{5.3}$$

where $\mathbf{V}_0(G)$ is the set of non-degenerate vertices, and $\mathbf{V}_\infty(G)$ is the set of degenerate vertices. Furthermore, each edge $x \in \mathbf{E}(G)$ connects a vertex in $\mathbf{V}_0(G)$ and a vertex in $\mathbf{V}_\infty(G)$ since $\mathbf{c}(x) \neq 0$. We may write $\mathbf{V}_\infty := \mathbf{V}_\infty(G)$ and $\mathbf{V}_0 := \mathbf{V}_0(G)$ when there is no confusion about G. Elements in $\mathbf{V}_0$ and in $\mathbf{V}_\infty$ are referred to as 0-*vertices* and ∞-*vertices* respectively.

Note that if $\boldsymbol{\tau}_{\Yup}$ is non-degenerate, then $\mathbf{V}_\infty(G) = \emptyset$ and $\mathbf{V}_0(G) = \mathbf{V}_{\max}(G)$ consists of a single vertex. On the other hand, when $\boldsymbol{\tau}_{\Yup}$ is degenerate, then $\mathbf{V}_{\max}(G) = \mathbf{V}_\infty(G)$ is the collection of degenerate vertices of G.

With this, we see that the data of $\boldsymbol{\tau}_{\Yup}$ is equivalent to a *decorated bipartite graph*

$$(G, \mathbf{V}(G) = \mathbf{V}_0 \sqcup \mathbf{V}_\infty, \mathbf{g}, \mathbf{deg}, \mathbf{m}, \mathbf{c}, \overline{\gamma}, \boldsymbol{\beta})\,. \tag{5.4}$$

Note that the information of $\boldsymbol{\sigma}$ and $\mathbf{V}_{\max}(G)$ may be recovered from the partition $\mathbf{V}(G) = \mathbf{V}_0 \sqcup \mathbf{V}_\infty$.

5.1.2. *Combinatorial decomposition.* For each $V \in \mathbf{V}(G)$, consider the graph G_V with

$$\mathbf{V}(G_V) = \{V\}, \qquad \mathbf{H}(G_V) = \mathbf{H}(V) = \{h \in \mathbf{H}(G) \mid \nu_G(h) = V\},$$

where the involution ι_{G_V} is the identity. Thus $\mathbf{H}(G_V) = \mathbf{L}(G_V)$ is the set of legs of G_V. For later use, denote by

$$\mathbf{L}_V = \mathbf{H}(G_V) \cap \mathbf{L}(G), \qquad \mathbf{S}_V := \mathbf{H}(G_V) \setminus \mathbf{L}_V.$$

Restricting the decorations in $\boldsymbol{\tau}_{\Yup}$ to G_V, we obtain a decorated type

$$\boldsymbol{\tau}_V = (\tau_V, \overline{\gamma}_V, \beta_V = \boldsymbol{\beta}(V_i)) \ \text{ with } \ \tau_V = (\mathbf{G}_V = (G_V, g_V = \mathbf{g}(V), \mathbf{deg}_V, \boldsymbol{\sigma}_V), \mathbf{c}_V)\,.$$

The compact type property of $\boldsymbol{\tau}_{\Yup}$ implies that

1. $\boldsymbol{\tau}_V$ is of compact type for any $V \in \mathbf{V}_\infty$.
2. For each $V \in \mathbf{V}_0$ and $h \in \mathbf{H}(G_V)$, $\mathbf{c}_V(h) > 0$ iff $h \in \mathbf{S}_V$. In particular, $\boldsymbol{\tau}_V$ is a decorated type of log R-maps.

For later use, we write $\mathbf{S}_0 = \sqcup_{V \in \mathbf{V}_0} \mathbf{S}_V$.

5.1.3. *Vertex virtual cycles.* For each $V \in \mathbf{V}(G)$, by (4.7) we have a Cartesian diagram with strict vertical arrows

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V) \arrow[r, "F_V"] \arrow[d, "\mathbf{ev}_V"] & \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \arrow[d, "\mathbf{ev}_V"] \\
\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_V) \arrow[r] & \mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V)
\end{tikzcd} \tag{5.5}$$

For convenience, we denote both vertical arrows by $\mathbf{ev}_V$.

Remark 5.1. For each $V \in \mathbf{V}_\infty$ observe that $\boldsymbol{\tau}_V$ and τ_V are realizable. Furthermore, there is a unique decorated $\curlywedge$-type $\boldsymbol{\tau}_{V,\curlywedge}$ (resp. $\curlywedge$-type $\tau_{V,\curlywedge}$) above $\boldsymbol{\tau}_V$ (resp. τ_V) obtained by setting $\mathbf{V}_{\max}(G_V) = \{V\}$. Indeed, any punctured map to $\mathcal{A}$ with tropical type τ_V is automatically a punctured map with the uniform maximal degeneracy of tropical $\curlywedge$-type $\tau_{V,\curlywedge}$.

By Remark 3.19 the three stacks $\mathfrak{M}(\mathcal{A}, \tau_V), \mathfrak{U}(\mathcal{A}, \tau_V)$ and $\mathfrak{U}(\mathcal{A}, \tau_{V,\curlywedge})$ share the same open dense substack parameterizing punctured maps with tropical type τ_V. By Corollary 3.17 we obtain $\mathfrak{U}(\mathcal{A}, \tau_V) = \mathfrak{U}(\mathcal{A}, \tau_{V,\curlywedge})$ hence $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V) = \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{V,\curlywedge})$.

By (2.56), the right vertical arrow in (5.5) admits a canonical perfect obstruction theory

$$\varphi_{\boldsymbol{\tau}_V,\mathrm{ev}} \colon \mathbb{T}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)/\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)} \longrightarrow \mathbb{E}_{\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)/\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)} =: \mathbb{E}_V \tag{5.6}$$

which further pulls back to the canonical perfect obstruction theory of the left vertical arrow. By Proposition B.28 this defines the canonical virtual cycles

$$[\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}}, \qquad [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}}.$$

By Proposition 3.13 and Remark 5.1, the bottom arrow in (5.5) is proper and birational. Thus we obtain a virtual push-forward

$$F_{V,*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} = [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}}.$$

If further $V \in \mathbf{V}_{\max}(G)$, then §4.5 provides a reduced perfect obstruction theory

$$\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_V,\mathrm{ev}} \colon \mathbb{T}_{\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_V)/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_V)} \longrightarrow \mathbb{E}_{\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_V)/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_V)} =: \mathbb{E}^{\mathrm{red}}_V.$$

By Proposition B.28 again, this defines the reduced virtual cycle $[\mathscr{U}(\boldsymbol{\tau}_V)]^{\mathrm{red}}$. For convenience, we write

$$[\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}} := F_{V,*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}}. \tag{5.7}$$

5.1.4. *Gluing of the underlying punctured R-maps.* For each $V \in \mathbf{V}(G)$ and each $h \in \mathbf{S}_V$, consider the evaluation $\mathrm{ev}_h \colon \mathscr{R}(\boldsymbol{\tau}_V) \to \overline{\gamma}_h$ given by (2.16). Taking products, we obtain

$$\mathrm{ev}_{\mathbf{S}} \colon \prod_{V \in \mathbf{V}(G)} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \longrightarrow \prod_V \prod_{h \in \mathbf{S}_V} \overline{\gamma}_h.$$

For each edge $x = \{h, \hat{h}\} \in \mathbf{E}(G)$ with $h \in \mathbf{S}_\infty$, set $\overline{\gamma}_x := \overline{\gamma}_h$, and consider the *involuted diagonal*

$$\Delta_x := Id \times \breve{\iota} \colon \overline{\gamma}_x \to \overline{\gamma}_h \times \overline{\gamma}_{\hat{h}} \tag{5.8}$$

where $\breve{\iota}$ is the nodal involution (2.17). Since $\mathcal{X}$ is smooth, Δ_x is a smooth closed embedding. Taking products, we obtain

$$\Delta_{\boldsymbol{\tau}_\curlywedge} := \prod_{x=\{h,\hat{h}\} \in \mathbf{E}(G)} \Delta_x \colon \prod_{x \in \mathbf{E}(G)} \overline{\gamma}_x \longrightarrow \prod_{\{h,\hat{h}\} \in \mathbf{E}(G)} \overline{\gamma}_h \times \overline{\gamma}_{\hat{h}} \cong \prod_{V \in \mathbf{V}(G)} \prod_{h \in \mathbf{S}_V} \overline{\gamma}_h. \tag{5.9}$$

Define $\mathscr{R}^G(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)$ via the Cartesian diagram

$$\begin{tikzcd}
\mathscr{R}^G(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge) \arrow[rr] \arrow[d] & & \prod_{V \in \mathbf{V}(G)} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \arrow[d, "\mathrm{ev}_{\mathbf{S}}"] \\
\prod_{x \in \mathbf{E}(G)} \overline{\gamma}_x \arrow[rr, "\Delta_{\boldsymbol{\tau}_\curlywedge}"'] & & \prod_{\{h, \hat{h}\} \in \mathbf{E}(G)} \overline{\gamma}_h \times \overline{\gamma}_{\hat{h}}
\end{tikzcd} \tag{5.10}$$

with strict horizontal arrows. Pulling back the universal punctured R-maps from each $\mathscr{R}(\tau_V)$, we obtain punctured R-maps over $\mathscr{R}^G(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)$ labeled by $\mathbf{V}(G)$. By §2.2.3 and the condition §2.3.1(i) on nodal sectors, the underlying R-map of these pull-back punctured R-maps glues to a stable underlying R-map

$$\underline{f}^G_{\boldsymbol{\tau}_\curlywedge} \colon \underline{C}^G_{\boldsymbol{\tau}_\curlywedge} \longrightarrow \underline{\mathfrak{P}}.$$

We thus define the tautological morphism

$$F^G_{\mathscr{M}} \colon \mathscr{R}^G(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge) \longrightarrow \mathscr{M}_{g,n}(\mathcal{X}, \beta)$$

by composing $\underline{f}^G_{\boldsymbol{\tau}_\curlywedge}$ with $\underline{\mathfrak{P}} \to \mathcal{X}$ and taking stablization.

Theorem 5.2. *Let $\boldsymbol{\tau}_\curlywedge$ be a decorated $\curlywedge$-type of rigid tropical curves as in (5.1). Further assume $\boldsymbol{\tau}_\curlywedge$ is of compact type. We have the following identity in* $\mathrm{CH}_*(\mathscr{M}_{g,n}(\mathcal{X}, \beta))$

$$\begin{aligned} F_{\mathscr{M},*}&[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)]^{\mathrm{red}} \\ &= (-\tilde{r})^{|\mathbf{V}_\infty(G)|-1} \mu(\boldsymbol{\tau}_\curlywedge) \cdot F^G_{\mathscr{M},*} \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{V \in \mathbf{V}_\infty} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}} \times \prod_{V \in \mathbf{V}_0} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right), \end{aligned} \tag{5.11}$$

where

$$\mu(\boldsymbol{\tau}_\curlywedge) = \frac{\prod_{E \in \mathbf{E}(G)} \mathbf{c}(E)}{\mathrm{lcm}_{E \in \mathbf{E}(G)} \mathbf{c}(E)}. \tag{5.12}$$

The proof of this theorem will be concluded in §5.4.

5.1.5. *The tropical decomposition formula for log GLSM.* Consider the decorated type $\boldsymbol{\tau}^\star = (\tau^\star, \overline{\gamma}, \boldsymbol{\beta})$ of log R-maps as in (4.41). Denote by

$$g = \mathbf{g}(\star), \qquad \beta = \boldsymbol{\beta}(\star), \qquad n = |\mathbf{L}(G^\star)|,$$

and consider the tautological morphism $F_{\mathscr{M}} \colon \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}^\star) \to \mathscr{M}_{g,n}(\mathcal{X}, \beta)$ as in (5.2).

Theorem 5.3. *Let $\boldsymbol{\tau}^\star = (\tau^\star, \overline{\gamma}, \boldsymbol{\beta})$ be a decorated type as in (4.41) consisting of a unique vertex, no edges, and that is of compact type. Then*

$$\begin{aligned} &F_{\mathscr{M},*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}^\star)]^{\mathrm{red}} = \\ &\sum_{\boldsymbol{\tau}_\curlywedge} (-\tilde{r})^{|\mathbf{V}_\infty|} \cdot \frac{\prod_{E \in \mathbf{E}(G)} \mathbf{c}(E)}{|\mathrm{Aut}(\tau_\curlywedge)|} \cdot F^G_{\mathscr{M},*} \Delta^!_{\boldsymbol{\tau}_\curlywedge} \left(\prod_{V \in \mathbf{V}_\infty} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}} \times \prod_{V \in \mathbf{V}_0} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right) \end{aligned} \tag{5.13}$$

where $\boldsymbol{\tau}_\curlywedge = (\tau_\curlywedge, \overline{\gamma}, \boldsymbol{\beta})$ runs through decorated $\curlywedge$-types of rigid tropical curves admitting contractions to $\boldsymbol{\tau}^\star$.

Remark 5.4. Note that $\boldsymbol{\tau}^\star$ is a decorated $\curlywedge$-type of rigid tropical curves with the construction $\boldsymbol{\tau}^\star \to \boldsymbol{\tau}^\star$ the identity, see §3.1.3. In this case, observe that $\mathbf{V}_\infty = \emptyset$, $|\mathrm{Aut}(\tau_\curlywedge)| = 1$, and $\mathbf{E}(G) = \emptyset$. Hence the summand corresponding to $\boldsymbol{\tau}^\star$ on the right hand side of the formula is naturally $F_{\mathscr{M},*}\left([\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{vir}}\right)$.

Proof. Recall from [18, Theorem 1.11] that

$$[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{red}} = [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{vir}} - \tilde{r}[\Delta^{\curlywedge}(\mathfrak{P}, \boldsymbol{\tau}^\star)]^{\mathrm{red}}.$$

Pushing forward along $F_{\mathscr{M}}$ and applying Theorem 4.15, we obtain

$$F_{\mathscr{M},*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{red}} = F_{\mathscr{M},*}\left([\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{vir}}\right) - \tilde{r}\sum_{\boldsymbol{\tau}_{\curlywedge}} \frac{\mathrm{lcm}_{x\in\mathbf{E}(G)}(\mathbf{c}(x))}{|\operatorname{Aut}(\tau_{\curlywedge})|} F_{\mathscr{M},*} i_{\boldsymbol{\tau}_{\curlywedge},*}[\mathscr{U}(\boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}}.$$

where $\boldsymbol{\tau}_{\curlywedge}$ runs through decorated realizable $\curlywedge$-types $\boldsymbol{\tau}_{\curlywedge} = (\tau_{\curlywedge}, \overline{\gamma}, \boldsymbol{\beta})$ of degenerate rigid tropical curves admitting contractions to $\boldsymbol{\tau}^\star$. Combining the above formula with Theorem 5.2, we obtain

$$\begin{aligned}(5.14)\quad F_{\mathscr{M},*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{red}} &= F_{\mathscr{M},*}\left([\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\star)]^{\mathrm{vir}}\right) \\ &+ \sum_{\boldsymbol{\tau}_{\curlywedge}} (-\tilde{r})^{|\mathbf{V}_\infty|} \cdot \frac{\prod_{E\in\mathbf{E}(G)}\mathbf{c}(E)}{|\operatorname{Aut}(\tau_{\curlywedge})|} \cdot F^{G}_{\mathscr{M},*}\Delta^{!}_{\mathbf{E}}\left(\prod_{V\in\mathbf{V}_\infty}[\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}} \times \prod_{V\in\mathbf{V}_0}[\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}}\right)\end{aligned}$$

noting that $e_{\max}(\tau_{\curlywedge})\mu(\boldsymbol{\tau}_{\curlywedge}) = \prod_{E\in\mathbf{E}(G)}\mathbf{c}(E)$ by (3.6) and (5.12). This finishes the proof. □

The rest of this section is devoted to the proof of Theorem 5.2, which splits into two major steps. We first, in §5.2, follow the gluing technique developed in [3] to separate contributions from the reduced theory of infinity. Unlike the situation in [3], since the reduced theory is only defined for stack of $\curlywedge$-punctured R-maps, the resulting formula involves contributions from possibly disconnected domain curves. In the second step, we further decompose infinity contributions to those from connected domain curves. We will briefly explain this step in §5.3, and postpone the details to §6.

5.2. **Splitting of punctured R-maps.**

5.2.1. *Virtual cycles of infinity.* Consider the graph $G_\infty = \sqcup_{V\in\mathbf{V}_\infty} G_V$. Observe that $\mathbf{H}(G_\infty) = \mathbf{L}(G_\infty)$ with a partition

$$\mathbf{H}(G_\infty) = \mathbf{L}_\infty \sqcup \mathbf{S}_\infty, \qquad \text{where } \ \mathbf{L}_\infty = \sqcup_{V\in\mathbf{V}_\infty} L_V, \ \ \mathbf{S}_\infty = \sqcup_{V\in\mathbf{V}_\infty}\mathbf{S}_V.$$

Restricting decorations of $\boldsymbol{\tau}_{\curlywedge}$ to G_∞, we obtain a decorated $\curlywedge$-type

$$\boldsymbol{\tau}_\infty = (\tau_\infty, \overline{\gamma}_\infty, \boldsymbol{\beta}_\infty, \mathbf{V}_{\max}(G_\infty) = \mathbf{V}(G_\infty))$$

where $\tau_\infty = (\mathbf{G}_\infty, \boldsymbol{\sigma}_\infty, \mathbf{c}_\infty)$ and $\mathbf{G}_\infty = (G_\infty, \mathbf{g}_\infty, \mathbf{deg}_\infty, \mathbf{m})$.

Since all vertices of G_∞ are degenerate, the fact that $\boldsymbol{\tau}_{\curlywedge}$ is of compact type implies that $\mathbf{c}_\infty(x) < 0$ for any $x \in \mathbf{L}(G_\infty)$. Hence $\boldsymbol{\tau}_\infty$ is of compact type. The decorated type of $\boldsymbol{\tau}_\infty$ obtained by removing the data $\mathbf{V}_{\max}(G_\infty)$ is the disjoint union $\sqcup_{V\in\mathbf{V}_\infty}\boldsymbol{\tau}_V$. By (4.7) we obtain a Cartesian diagram

$$(5.15)\qquad \begin{tikzcd} \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\infty) \arrow[r, "\mathrm{spl}_{\mathscr{R}}"] \arrow[d, "\mathbf{ev}_\infty"'] & \prod_{V\in\mathbf{V}_\infty}\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \arrow[d, "\prod_V \mathbf{ev}_V"] \\ \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty) \arrow[r] & \prod_{V\in\mathbf{V}_\infty}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V) \end{tikzcd}$$

with both vertical arrows strict. The strict arrow $\mathbf{ev}_\infty$ admits a canonical perfect obstruction theory

$$(5.16)\qquad \varphi_\infty\colon \mathbb{T}_{\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)} \longrightarrow \mathbb{E}_{\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)} =: \mathbb{E}_\infty$$

by pulling back (2.56), as well as a reduced perfect obstruction theory, as in §4.6

$$(5.17)\qquad \varphi^{\mathrm{red}}_\infty\colon \mathbb{T}_{\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)} \longrightarrow \mathbb{E}^{\mathrm{red}}_{\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)/\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)} =: \mathbb{E}^{\mathrm{red}}_\infty$$

By Proposition 3.18 and the fact that both τ_∞ and $\sqcup_{V\in\mathbf{V}_\infty}\boldsymbol{\tau}_V$ are realizable, the two theories φ_∞ and $\varphi_\infty^{\mathrm{red}}$ lead to the canonical and reduced virtual cycles

$$[\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)]^{\mathrm{vir}}, \qquad [\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)]^{\mathrm{red}},$$

respectively, which are related via Corollary 4.14.

5.2.2. *Splitting punctured maps with marked domains.* Let $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ be a punctured map over an fs base W. Suppose that the domain punctured curve $\pi\colon C^\circ \to W$ is marked by a decorated graph $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ as in (5.1). For each $x \in \mathbf{E}(G)$, let $p_x \subset C^\circ$ be the corresponding node. Taking the partial normalization along $\sqcup_{x\in\mathbf{E}(G)}p_x$, we obtain the strict morphism

$$\widetilde{C^\circ} := \bigsqcup_{V\in\mathbf{V}(G)} \widetilde{C^\circ_V} \longrightarrow C^\circ \tag{5.18}$$

where $\widetilde{C^\circ_V}$ is the component labeled by the vertex $V \in \mathbf{V}(G)$. Consider the morphisms

$$\widetilde{\mathfrak{f}}\colon \widetilde{C^\circ} \to \mathcal{A}, \qquad \widetilde{\pi}\colon \widetilde{C^\circ} \to W$$

obtained by composing (5.18) with $\mathfrak{f}$ and with π respectively. By [3, Prop. 5.2], $\widetilde{\pi}\colon \widetilde{C^\circ} \to W$ is a family of punctured curves over W, hence $\widetilde{\mathfrak{f}}$ is a family of punctured maps over W, but not necessarily pre-stable in general. Indeed, for each edge $x = \{h, \hat{h}\} \in \mathbf{E}(G)$, denote by $\widetilde{p}_h, \widetilde{p}_{\hat{h}} \subset \widetilde{C^\circ}$ the pre-images of p_x. The punctured map $\widetilde{\mathfrak{f}}$ fails to be pre-stable along $\widetilde{p}_h$ and $\widetilde{p}_{\hat{h}}$ in general, see §B.2.7.

For each vertex V, denote by $\widetilde{\mathfrak{f}}_V := \widetilde{\mathfrak{f}}|_{\widetilde{C^\circ_V}}\colon \widetilde{C^\circ_V} \to \mathcal{A}$ the punctured map over W obtained by restriction. Applying the pre-stabilization construction of [3, Prop. 2.5], we obtain a canonical factorization

$$\begin{tikzcd} \widetilde{C^\circ_V} \arrow[rr, "\widetilde{\mathfrak{f}}_V"] \arrow[dr, "\mathrm{P}"'] & & \mathcal{A} \\ & C^\circ_V \arrow[ur, "\mathfrak{f}_V"'] & \end{tikzcd} \tag{5.19}$$

which satisfies the following properties

1. (1) P is a morphism of fine log schemes such that $\underline{\mathrm{P}}$ is the identity of the underlying pre-stable curve, $\mathrm{P}^\flat$ induces an isomorphism $(\mathrm{P}^\flat)^{gp}\colon \mathcal{M}^{gp}_{\widetilde{C^\circ_V}} \to \mathcal{M}^{gp}_{C^\circ_V}$, and the restriction $\mathrm{P}|_{\widetilde{C^\circ_V}\setminus\sqcup_{h\in\mathbf{S}_V}\widetilde{p}_h}$ is the identity of log curves.
2. (2) $\mathfrak{f}_V$ is a pre-stable punctured map over W with the underlying $\underline{\mathfrak{f}_V} = \underline{\widetilde{\mathfrak{f}}_V}$, and
$$\widetilde{\mathfrak{f}}_V|_{\widetilde{C^\circ_V}\setminus\sqcup_{h\in\mathbf{S}_V}\widetilde{p}_h} = \mathfrak{f}_V|_{C^\circ_V\setminus(\sqcup_{h\in\mathbf{S}_V}\widetilde{p}_h)},$$
where $p_h \subset C^\circ_V$ is the gerbe corresponding to $h \in \mathbf{S}_V$ from a splitting edge.

It follows that the two punctured maps $\mathfrak{f}_{\nu(h)}$ and $\mathfrak{f}$ have the same contact order along any half-edge $h \in \mathbf{H}(G)$.

5.2.3. *Splitting punctured maps over* $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\curlywedge)$. We apply the discussion of §5.2.2 to the universal family

$$\begin{tikzcd} \sqcup_{h\in\mathbf{H}(G)}\mathfrak{p}_h \arrow[r] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\ \mathfrak{C}^\circ(\tau_\curlywedge) \arrow[r, "\mathfrak{f}_{\tau_\curlywedge}"] & \mathcal{A} \end{tikzcd} \tag{5.20}$$

over $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\curlywedge})$, with the projection $\pi_{\tau_{\curlywedge}}\colon \mathfrak{C}^{\circ}(\tau_{\curlywedge}) \longrightarrow \mathfrak{U}^{\mathrm{ev}}(\tau_{\curlywedge})$. We obtain a (not necessarily pre-stable) punctured map $\widetilde{\mathfrak{f}_{\tau_{\curlywedge}}}$ over W given by the composition

$$\widetilde{\mathfrak{f}_{\tau_{\curlywedge}}} := \bigsqcup_{V\in\mathbf{V}(G)} \widetilde{\mathfrak{f}_{\tau_{\curlywedge},V}}\colon \widetilde{\mathfrak{C}^{\circ}(\tau_{\curlywedge})} = \bigsqcup_{V\in\mathbf{V}(G)} \widetilde{\mathfrak{C}^{\circ}_{\tau_{\curlywedge},V}} \longrightarrow \mathfrak{C}^{\circ}(\tau_{\curlywedge}) \xrightarrow{\mathfrak{f}_{\tau_{\curlywedge}}} \mathcal{A} \tag{5.21}$$

where the left arrow is the partial normalization along nodes $\sqcup_{x\in\mathbf{E}(G)}\mathfrak{p}_x$, and $\widetilde{\mathfrak{f}_{\tau_{\curlywedge},V}}$ is the restriction to the component $\widetilde{\mathfrak{C}^{\circ}_{\tau_{\curlywedge},V}}$ corresponding to V.

Let $\mathfrak{f}_{\tau_{\curlywedge},V}\colon \mathfrak{C}^{\circ}_{\tau_{\curlywedge},V} \to \mathcal{A}$ be the punctured map over $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\curlywedge})$ obtained by taking the pre-stabilization of $\widetilde{\mathfrak{f}_{\tau_{\curlywedge},V}}$ as in (5.19). Since pre-stablizating does not change contact orders nor the underlying structures, for each $V\in\mathbf{V}(G)$ the component $\mathfrak{f}_{\tau_{\curlywedge},V}$ is a punctured map marked by τ_V fitting into a commutative diagram induced by (5.20):

$$\begin{tikzcd}
\bigsqcup_{h\in\mathbf{H}(G_V)}\mathfrak{p}_h \arrow[r] \arrow[d] & \bigsqcup_{h\in\mathbf{H}(G_V)}\gamma_h \arrow[d] \\
\mathfrak{C}^{\circ}_{\tau_{\curlywedge},V} \arrow[r, "\mathfrak{f}_{\tau_{\curlywedge},V}"] & \mathcal{A}
\end{tikzcd} \tag{5.22}$$

On the other hand, consider the punctured map

$$\mathfrak{f}_{\tau_{\curlywedge},\infty} := \bigsqcup_{V\in\mathbf{V}_{\infty}} \mathfrak{f}_{\tau_{\curlywedge},V}\colon \mathfrak{C}^{\circ}_{\tau_{\curlywedge},\infty} := \bigsqcup_{V\in\mathbf{V}_{\infty}} \mathfrak{C}^{\circ}_{\tau_{\curlywedge},V} \longrightarrow \mathcal{A}. \tag{5.23}$$

As the pre-stablization only modifies punctured maps along gerbes from splitting nodes, we further observe that $\mathfrak{f}_{\tau_{\curlywedge},\infty}$ is a $\curlywedge$-punctured map over $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\curlywedge})$ marked by τ_{∞}, hence we have the commutative diagram

$$\begin{tikzcd}
\bigsqcup_{h\in\mathbf{H}(G_{\infty})}\mathfrak{p}_h \arrow[r] \arrow[d] & \bigsqcup_{h\in\mathbf{H}(G_{\infty})}\gamma_h \arrow[d] \\
\mathfrak{C}^{\circ}_{\tau_{\curlywedge},\infty} \arrow[r, "\mathfrak{f}_{\tau_{\curlywedge},\infty}"] & \mathcal{A}.
\end{tikzcd} \tag{5.24}$$

We thus obtain a sequence of tautological morphisms

$$\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\curlywedge}) \xrightarrow{\mathfrak{spl}_{\infty|0}} \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty}) \times \prod_{V\in\mathbf{V}_0}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \longrightarrow \prod_{V\in\mathbf{V}(G)}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \tag{5.25}$$

where the first arrow is given by (5.24) and by (5.22) for each $V\in\mathbf{V}_0$, and the second arrow is induced by (5.22) for all $V\in\mathbf{V}(G)$.

5.2.4. *Splitting punctured R-maps over $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$.* Consider the universal family

$$f_{\boldsymbol{\tau}_{\curlywedge}}\colon C^{\circ}_{\boldsymbol{\tau}_{\curlywedge}} \to \mathfrak{P}, \qquad \pi_{\boldsymbol{\tau}_{\curlywedge}}\colon C^{\circ}_{\boldsymbol{\tau}_{\curlywedge}} \to \mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$$

over $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$. Let $\mathfrak{f}_{\boldsymbol{\tau}_{\curlywedge}}\colon C^{\circ}_{\boldsymbol{\tau}_{\curlywedge}} \to \mathcal{A}$ be the associated punctured map to $\mathcal{A}$. Similar to (5.21), we may again take partial normalization along nodes $\sqcup_{x\in\mathbf{E}(G)}p_x \subset C^{\circ}_{\boldsymbol{\tau}_{\curlywedge}}$ and composing with $f_{\boldsymbol{\tau}_{\curlywedge}}$ to obtain a (not necessarily pre-stable) punctured R-map

$$\widetilde{f_{\boldsymbol{\tau}_{\curlywedge}}} := \bigsqcup_{V\in\mathbf{V}(G)} \widetilde{f_{\boldsymbol{\tau}_{\curlywedge},V}}\colon \widetilde{C^{\circ}_{\boldsymbol{\tau}_{\curlywedge}}} = \bigsqcup_{V\in\mathbf{V}(G)} \widetilde{C^{\circ}_{\boldsymbol{\tau}_{\curlywedge},V}} \longrightarrow C^{\circ}_{\boldsymbol{\tau}_{\curlywedge}} \xrightarrow{f_{\boldsymbol{\tau}_{\curlywedge}}} \mathfrak{P}. \tag{5.26}$$

Pre-stablizing $\widetilde{f_{\boldsymbol{\tau}_{\curlywedge},V}}$ as in (5.19) using [3, Prop. 2.5], we obtain a stable punctured R-map

$$f_{\boldsymbol{\tau}_{\curlywedge},V}\colon C^{\circ}_{\boldsymbol{\tau}_{\curlywedge},V} \longrightarrow \mathfrak{P} \tag{5.27}$$

over $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$, marked by $\boldsymbol{\tau}_V$.

Indeed the universal punctured R-map $f_{\boldsymbol{\tau}_\curlywedge}$ induces a commutative diagram

$$\begin{tikzcd}
\sqcup_{h\in\mathbf{H}(G)} p_h \arrow[r] \arrow[d] & \mathfrak{P}_{\mathbf{k}} \arrow[d] \\
C^\circ_{\boldsymbol{\tau}_\curlywedge} \arrow[r, "\mathfrak{f}_{\tau_\curlywedge}"] & \mathcal{A}
\end{tikzcd} \tag{5.28}$$

over $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\curlywedge)$. This defines the tautological morphism $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\curlywedge)\to\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\curlywedge)$ as in (4.7), along which (5.20) pulls back to (5.28). For each vertex V, pulling back the universal splitting (5.22), we obtain a commutative diagrams

$$\begin{tikzcd}
\bigsqcup_{h\in\mathbf{H}(G_V)} p_h \arrow[r] \arrow[d] & \bigsqcup_{h\in\mathbf{H}(G_V)} \gamma_h \arrow[d] \\
C^\circ_{\boldsymbol{\tau}_\curlywedge,V} \arrow[r, "\mathfrak{f}_{\tau_\curlywedge,V}"] & \mathcal{A}
\end{tikzcd} \tag{5.29}$$

Since $\underline{\mathfrak{f}_{\tau_\curlywedge,V}} = \underline{\widetilde{f_{\boldsymbol{\tau}_\curlywedge,V}}}$, we observe that (5.29) lifts to the punctured R-map $f_{\boldsymbol{\tau}_\curlywedge,V}$. Further observe that the punctured map

$$f_{\boldsymbol{\tau}_\curlywedge,\infty} := \bigsqcup_{V\in\mathbf{V}_\infty} f_{\boldsymbol{\tau}_\curlywedge,V}\colon C^\circ_{\boldsymbol{\tau}_\curlywedge,\infty} := \bigsqcup_{V\in\mathbf{V}_\infty} C^\circ_{\boldsymbol{\tau}_\curlywedge,V} \longrightarrow \mathfrak{P} \tag{5.30}$$

is the lift of (5.23), hence is a $\curlywedge$-punctured map over $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\curlywedge)$ marked by $\boldsymbol{\tau}_\infty$.

Similar to (5.25), we have a sequence of tautological morphisms

$$\mathscr{U}^{\mathrm{ev}}(\mathfrak{P},\boldsymbol{\tau}_\curlywedge) \xrightarrow{\mathrm{spl}_{\infty|0}} \mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)\times\textstyle\prod_{V\in\mathbf{V}_0}\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V) \longrightarrow \textstyle\prod_{V\in\mathbf{V}(G)}\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V) \tag{5.31}$$

induced by (5.27) and (5.30).

Proposition 5.5. *There is a canonical Cartesian diagram*

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\curlywedge) \arrow[r, "\mathrm{spl}_{\infty|0}"] \arrow[d, "\mathbf{ev}"'] & \mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)\times\prod_{V\in\mathbf{V}_0}\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V) \arrow[d, "\widetilde{\mathbf{ev}}:=\mathbf{ev}_\infty\times\prod\mathbf{ev}_V"] \\
\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\curlywedge) \arrow[r, "\mathfrak{spl}_{\infty|0}"] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)\times\prod_{V\in\mathbf{V}_0}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)
\end{tikzcd} \tag{5.32}$$

with strict vertical arrows defined as in (4.7)*.*

Proof. This is identical to [3, Prop. 5.17]. Indeed as the pre-stabilization of [3, Prop. 2.5] does not modify the underlying structures, the splitting of punctured R-maps is obtained by splitting the associated punctured maps to $\mathcal{A}$ as explained above. ☐

5.2.5. *Perfect obstruction theories along splittings.* Taking the product of (5.6) and (5.16), we obtain the perfect obstruction theory

$$\varphi_{\widetilde{\mathbf{ev}}}\colon \mathbb{T}_{\widetilde{\mathbf{ev}}} \longrightarrow \mathbb{E}_\infty\oplus\bigoplus_{V\in\mathbf{V}_0}\mathbb{E}_V := \mathbb{E}_{\widetilde{\mathbf{ev}}} \tag{5.33}$$

of the morphism $\widetilde{\mathbf{ev}}$ in (5.32). Similarly taking the product of (5.6) and (5.17), we obtain another perfect obstruction theory of $\widetilde{\mathbf{ev}}$

$$\varphi^{\mathrm{red}}_{\widetilde{\mathbf{ev}}}\colon \mathbb{T}_{\widetilde{\mathbf{ev}}} \longrightarrow \mathbb{E}^{\mathrm{red}}_\infty\oplus\bigoplus_{V\in\mathbf{V}_0}\mathbb{E}_V := \mathbb{E}^{\mathrm{red}}_{\widetilde{\mathbf{ev}}}. \tag{5.34}$$

Proposition 5.6. *(1) The canonical theory $\varphi_{\mathbf{ev}}$ of $\mathbf{ev}$ coincide with the perfect obstruction theory by pulling back $\varphi_{\widetilde{\mathbf{ev}}}$.*

(2) The reduced theory $\varphi^{\mathrm{red}}_{\mathbf{ev}}$ of $\mathbf{ev}$ coincide with the perfect obstruction theory by pulling back $\varphi^{\mathrm{red}}_{\widetilde{\mathbf{ev}}}$.

Proof. We write for simplicity

$$\mathscr{U}_{\infty|0} := \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\infty) \times \prod_{V \in \mathbf{V}_0} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V), \qquad \mathfrak{U}^{\mathrm{ev}}_{\infty|0} := \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty) \times \prod_{V \in \mathbf{V}_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V)$$

and denote by

$$f_{\infty|0} := f_{\boldsymbol{\tau}_\infty} \sqcup \bigsqcup_{V \in \mathbf{V}_0} f_{\boldsymbol{\tau}_V} : C^\circ_{\infty|0} := C^\circ_{\boldsymbol{\tau}_\infty} \sqcup \bigsqcup_{V \in \mathbf{V}_0} C^\circ_{\boldsymbol{\tau}_V} \longrightarrow \mathfrak{P}$$

the universal punctured R-map over $\mathscr{U}_{\infty|0}$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\curlywedge) \arrow[d, "\mathfrak{spl}_{\infty|0}"'] & \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge) \arrow[l, "\mathbf{ev}"'] \arrow[d, "\mathrm{spl}_{\infty|0}"'] & C^\circ_{\boldsymbol{\tau}_\curlywedge} \arrow[l, "\pi_{\boldsymbol{\tau}_\curlywedge}"'] \arrow[rr, bend left, "f_{\boldsymbol{\tau}_\curlywedge}"] & \widetilde{C^\circ_{\boldsymbol{\tau}_\curlywedge}} \arrow[l] \arrow[r, "\widetilde{f_{\boldsymbol{\tau}_\curlywedge}}"'] \arrow[d] \arrow[ll, bend left, "\widetilde{\pi_{\boldsymbol{\tau}_\curlywedge}}"] & \mathfrak{P} \\
\mathfrak{U}^{\mathrm{ev}}_{\infty|0} & \mathscr{U}_{\infty|0} \arrow[l, "\widetilde{\mathbf{ev}}"] & & C^\circ_{\infty|0} \arrow[ll, "\pi_{\infty|0}"] \arrow[ur, "f_{\infty|0}"'] &
\end{tikzcd}$$

Recall from (2.56) that we have the canonical perfect obstruction theories

$$\begin{aligned} \varphi_{\mathbf{ev}} \colon \mathbb{T}_{\mathbf{ev}} &\longrightarrow \mathbb{E}_{\mathbf{ev}} := R(\widetilde{\pi_{\boldsymbol{\tau}_\curlywedge}})_* \widetilde{f_{\boldsymbol{\tau}_\curlywedge}}^{*}(T_{\mathfrak{P}/\mathbf{BC}^*_\omega}) \\ \varphi_{\widetilde{\mathbf{ev}}} \colon \mathbb{T}_{\widetilde{\mathbf{ev}}} &\longrightarrow \mathbb{E}_{\widetilde{\mathbf{ev}}} := R(\pi_{\infty|0})_* f^*_{\infty|0}(T_{\mathfrak{P}/\mathbf{BC}^*_\omega}) \end{aligned}$$

for $\mathbf{ev}$ and $\widetilde{\mathbf{ev}}$ respectively. On the level of underlying stacks note that

$$\underline{\widetilde{C^\circ_{\boldsymbol{\tau}_\curlywedge}}} \cong \underline{C^\circ_{\infty|0}} \times_{\underline{\mathscr{U}_{\infty|0}}} \underline{\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)}.$$

Thus flat base change implies $\mathrm{spl}^*_{\infty|0} \mathbb{E}_{\widetilde{\mathbf{ev}}} \cong \mathbb{E}_{\mathbf{ev}}$. By [3, Thm. 5.19 (1)], this isomorphism fits in a commutative diagram

(5.35)
$$\begin{tikzcd}
\mathbb{T}_{\mathbf{ev}} \arrow[r] \arrow[d, "\varphi_{\mathbf{ev}}"'] & \mathrm{spl}^*_{\infty|0} \mathbb{T}_{\widetilde{\mathbf{ev}}} \arrow[d, "\mathrm{spl}^*_{\infty|0} \varphi_{\widetilde{\mathbf{ev}}}"] \\
\mathbb{E}_{\mathbf{ev}} \arrow[r, equal] & \mathrm{spl}^*_{\infty|0} \mathbb{E}_{\widetilde{\mathbf{ev}}}
\end{tikzcd}$$

In particular, the perfect obstruction theory obtained by pulling back $\varphi_{\widetilde{\mathbf{ev}}}$ coincides with the canonical one $\varphi_{\mathbf{ev}}$. This proves (1).

To further prove the pull-back property of the reduced theory, we observe the commutativity of

(5.36)
$$\begin{tikzcd}
& \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\curlywedge) \arrow[dl, bend right] \arrow[d, "\mathfrak{spl}_{\infty|0}"] & \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge) \arrow[l, "\mathbf{ev}"'] \arrow[d, "\mathrm{spl}_{\infty|0}"] \\
\mathcal{A}_{\max} & \mathfrak{U}^{\mathrm{ev}}_{\infty|0} \arrow[l] & \mathscr{U}_{\infty|0} \arrow[l, "\widetilde{\mathbf{ev}}"]
\end{tikzcd}$$

where the arrow $\mathfrak{U}^{\mathrm{ev}}_{\infty|0} \to \mathcal{A}_{\max}$ is given by the projection $\mathfrak{U}^{\mathrm{ev}}_{\infty|0} \to \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty)$. Recall the boundary complex $\mathbb{F} := \mathbf{L}^{-\otimes \tilde{r}}_{\max}[-1]$ from §4.4. We obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{T}_{\mathbf{ev}} \arrow[d] \arrow[dr, "\varphi^{\mathrm{red}}_{\mathbf{ev}}"] & & & & \\
\mathrm{spl}^*_{\infty|0} \mathbb{T}_{\widetilde{\mathbf{ev}}} \arrow[dr, "\varphi^{\mathrm{red}}_{\widetilde{\mathbf{ev}}}"'] & \mathbb{E}^{\mathrm{red}}_{\mathbf{ev}} \arrow[r] \arrow[d, "\cong"] & \mathbb{E}_{\mathbf{ev}} \arrow[r] \arrow[d, "\cong"] & \mathbb{F}|_{\mathscr{U}_{\infty|0}} \arrow[r, "{[1]}"] \arrow[d, "\cong"] & {} \\
& \mathrm{spl}^*_{\infty|0} \mathbb{E}^{\mathrm{red}}_{\widetilde{\mathbf{ev}}} \arrow[r] & \mathrm{spl}^*_{\infty|0} \mathbb{E}_{\widetilde{\mathbf{ev}}} \arrow[r] & \mathrm{spl}^*_{\infty|0}(\mathbb{F}|_{\mathscr{U}_{\infty|0}}) \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

as follows. First, the the two horizontal rows are two distinguished triangles from (4.35). The commutativity of (5.36) implies the right vertical arrow is an isomorphism, and the square on the right is commutative. The commutativity of the square on the right hand side is from the commutativity of (5.36). The isomorphism $\mathbb{E}^{\mathrm{red}}_{\mathbf{ev}} \to \mathrm{spl}^*_{\infty|0}\mathbb{E}_{\widetilde{\mathbf{ev}}}$ follows from the construction of the canonical theory, see (2.55). Hence the two distinguished triangles are isomorphic. Further applying the commutativity of (5.35), we obtain the commutativity of the skewed diamond on the left hand side, finishing the proof of (2). ☐

5.2.6. *A factorization of* $\mathfrak{spl}_{\infty|0}$. Consider the products of evaluations

$$\begin{aligned}
\mathrm{ev}_{\mathbf{S}_V} &:= \prod_{h\in\mathbf{S}_V} \mathrm{ev}_h \colon \mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \to \overline{\boldsymbol{\gamma}}_{\mathbf{S}_V} := \prod_{h\in\mathbf{S}_V} \overline{\boldsymbol{\gamma}}_h \\
\mathrm{ev}_{\mathbf{S}_\infty} &:= \prod_{h\in\mathbf{S}_\infty} \mathrm{ev}_h \colon \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty) \to \overline{\boldsymbol{\gamma}}_{\mathbf{S}_\infty} := \prod_{h\in\mathbf{S}_\infty} \overline{\boldsymbol{\gamma}}_h \\
\mathrm{ev}_{\mathbf{E}} &:= \prod_{h\in\mathbf{S}_\infty} \mathrm{ev}_h \colon \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\curlywedge) \to \overline{\boldsymbol{\gamma}}_{\mathbf{S}_\infty}
\end{aligned}$$

where ev_h is given by the orbifold structure along marking or nodal gerbes as in (2.16). Following the notations in (5.8), we have $\overline{\boldsymbol{\gamma}}_{\mathbf{S}_\infty} = \prod_{x\in\mathbf{E}(G)} \overline{\boldsymbol{\gamma}}_x$.

Proposition 5.7. *We obtain a commutative diagram*

$$\begin{tikzcd}
\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\curlywedge) \arrow[r, "\mathfrak{spl}_{\log}"'] \arrow[rr, bend left, "\mathfrak{spl}_{\infty|0}"] \arrow[dr, bend right, "\mathrm{ev}_{\mathbf{E}}"'] & \mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_\curlywedge) \arrow[r, "\mathfrak{spl}_G"'] \arrow[d, "\mathrm{ev}^G_{\mathbf{E}}"'] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty) \times \prod_{V\in\mathbf{V}_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \arrow[d, "\mathrm{ev}_{\mathbf{S}}"] \\
 & \prod_{x\in\mathbf{E}(G)} \overline{\boldsymbol{\gamma}}_x \arrow[r, "\Delta_{\boldsymbol{\tau}_\curlywedge}"'] & \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)} \overline{\boldsymbol{\gamma}}_h \times \overline{\boldsymbol{\gamma}}_{\hat{h}}
\end{tikzcd} \tag{5.37}$$

where $\mathrm{ev}_{\mathbf{S}} := \mathrm{ev}_{\mathbf{S}_\infty} \times \prod \mathrm{ev}_{\mathbf{S}_{V_i}}$, $\Delta_{\boldsymbol{\tau}_\curlywedge}$ *is given by* (5.9), *and the square is Cartesian.*

Proof. Pulling back universal families over $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)$ and $\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)$, we obtain families of punctured maps with evaluations along markings over $\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_\curlywedge)$ respectively:

$$\begin{tikzcd}
\sqcup_{x\in\mathbf{H}(G_\infty)}\mathfrak{p}_x \arrow[r] \arrow[d] & \sqcup_{x\in\mathbf{H}(G_\infty)}\boldsymbol{\gamma}_{\infty,x} \arrow[d] & \text{and} & \sqcup_{x\in\mathbf{H}(G_V)}\mathfrak{p}_x \arrow[r] \arrow[d] & \sqcup_{x\in\mathbf{H}(G_V)}\boldsymbol{\gamma}_{\infty,x} \arrow[d] \\
\mathfrak{C}^\circ_{\infty,G} \arrow[r, "\mathfrak{f}^G_\infty"] & \mathcal{A}, & & \mathfrak{C}^\circ_{V,G} \arrow[r, "\mathfrak{f}^G_V"] & \mathcal{A}
\end{tikzcd} \tag{5.38}$$

The Cartesian square further implies a commutative diagram of underlying stacks

$$\begin{tikzcd}
\sqcup_{x\in\mathbf{L}(G)\sqcup\mathbf{E}(G)}\underline{\mathfrak{p}_x} \arrow[r] \arrow[d] & \sqcup_{x\in\mathbf{L}(G)\sqcup\mathbf{E}(G)}\underline{\boldsymbol{\gamma}_x} \arrow[d] \\
\underline{\mathfrak{C}_G} \arrow[r, "\underline{\mathfrak{f}}^G"] & \underline{\mathcal{A}}
\end{tikzcd} \tag{5.39}$$

where $\underline{\mathfrak{C}_G}$ is an underlying curve marked by $(G,\mathbf{g},\mathbf{deg})$ obtained by gluing underlying curves

$$\underline{\mathfrak{C}_G} = \underline{\mathfrak{C}^\circ_{\infty,G}} \cup \bigcup_{V\in\mathbf{V}_0} \underline{\mathfrak{C}^\circ_{V,G}}, \quad \text{with } \underline{\mathfrak{p}_{\hat{h}}} \xleftarrow{\iota} \underline{\mathfrak{p}_x} \xrightarrow{\cong} \underline{\mathfrak{p}_h} \text{ for all } x=\{h,\hat{h}\}\in\mathbf{E}(G)$$

where $\underline{\mathfrak{p}_{\hat{h}}} \subset \bigcup_V \underline{\mathfrak{C}^\circ_{V,G}}$ and $\underline{\mathfrak{p}_h} \subset \underline{\mathfrak{C}^\circ_{\infty,G}}$ are gerbes corresponding to $\hat{h}$ and h, and $\underline{\mathfrak{p}_x} \subset \underline{\mathfrak{C}_G}$ is the nodal gerbe of x. The factorization of the left triangle in (5.37) follows from the fact that the evaluations $\mathbf{ev}_{\mathbf{E}}$ and $\mathbf{ev}^G_{\mathbf{E}}$ are both induced by the gerbes $\underline{\mathfrak{p}_x} \cong \underline{\mathfrak{p}_h}$ for $h\in\mathbf{S}_\infty$.

The morphism $\mathfrak{spl}_{\log}$ is then the tautological morphisms such that (5.38) pulls back to the two families (5.22) and (5.23), whose underlying families glue to the underlying of (5.20). In particular, we obtain the factorization $\mathfrak{spl}_{\infty|0} = \mathfrak{spl}_G \circ \mathfrak{spl}_{\log}$ as needed. ☐

Proposition 5.8. *The morphism* $\mathfrak{spl}_{\log}$ *in* (5.37) *is proper, representable and generically finite of degree* $\mu(\boldsymbol{\tau}_{\curlywedge})$ *as in* (5.12)*.*

Because the proof of this proposition requires a discussion on gluing punctured maps, we will postpone it to §5.5.

5.2.7. *The gluing formula of reduced virtual cycles.* Combining (5.32) and (5.37), we obtain a commutative diagram with all squares Cartesian in the fs category

(5.40)
$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge}) \arrow[r, "\mathrm{spl}_{\log}"'] \arrow[rr, bend left, "\mathrm{spl}_{\infty|0}"] \arrow[d, "\mathbf{ev}"'] & \mathscr{U}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge}) \arrow[r, "\mathrm{spl}_G"'] \arrow[d, "\mathbf{ev}^G_{\mathbf{E}}"] & \mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\infty}) \times \prod_{V\in\mathbf{V}_0} \mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V) \arrow[d, "\widetilde{\mathbf{ev}}"] \\
\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\curlywedge}) \arrow[r, "\mathfrak{spl}_{\log}"'] & \mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\curlywedge}) \arrow[r, "\mathfrak{spl}_G"'] \arrow[d, "\mathrm{ev}^G_{\mathbf{E}}"] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty}) \times \prod_{V\in\mathbf{V}_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \arrow[d, "\mathrm{ev}_{\mathbf{S}}"] \\
 & \prod_{x\in\mathbf{E}(G)} \overline{\gamma}_x \arrow[r, "\Delta_{\boldsymbol{\tau}_{\curlywedge}}"'] & \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)} \overline{\gamma}_h \times \overline{\gamma}_{\hat{h}}
\end{tikzcd}$$

The stack $\mathscr{U}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})$ parameterizes punctured R-maps marked by $\boldsymbol{\tau}_\infty$ and $\boldsymbol{\tau}_V$ for $V \in \mathbf{V}_0$ with their underlying R-maps glued along $\mathbf{E}(G)$. Note that the three vertical arrows $\mathbf{ev}, \mathbf{ev}^G_{\mathbf{E}}$ and $\widetilde{\mathbf{ev}}$ are all strict, hence the two upper squares are both Cartesian of the underlying stacks.

By pulling back (5.34), we obtain a perfect obstruction theory

$$\varphi^{\mathrm{red}}_{\mathbf{ev}^G_{\mathbf{E}}} \colon \mathbb{T}_{\mathbf{ev}^G_{\mathbf{E}}} \longrightarrow \mathbb{E}^{\mathrm{red}}_{\mathbf{ev}^G_{\mathbf{E}}} = \mathrm{spl}^*_G \, \mathbb{E}^{\mathrm{red}}_{\widetilde{\mathbf{ev}}}.$$

of $\mathbf{ev}^G_{\mathbf{E}}$. Note that $\mathrm{ev}_{\mathbf{S}}$ is a fibration, hence is flat. Proposition 3.18 and Proposition B.28 imply that $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty) \times \prod_{V\in bV_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)$ is pure-dimensional. Hence $\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\curlywedge})$ is also pure-dimensional, defining the fundamental class $[\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\curlywedge})]$. Let $\mathbf{ev}^{G,!}_{\mathbf{E},\mathrm{red}}$ be the virtual pull-back of $\mathbf{ev}^G_{\mathbf{E}}$ defined by $\varphi^{\mathrm{red}}_{\mathbf{ev}_G}$. We obtain a virtual cycle

$$[\mathscr{U}^G(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}} := \mathbf{ev}^{G,!}_{\mathbf{E},\mathrm{red}}[\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\curlywedge})].$$

Since $\Delta_{\mathbf{E}}$ is a locally complete intersection, we have the Gysn pull-back

$$\Delta^!_{\mathbf{E}}[\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty) \times \prod_{V\in\mathbf{V}_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)] = [\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\curlywedge})]. \tag{5.41}$$

Denote by $[\Delta_{\boldsymbol{\tau}_{\curlywedge}}] \subset \mathrm{CH}^*(\prod_{\{h,\hat{h}\}\in\mathbf{E}(G)} \overline{\gamma}_h \times \overline{\gamma}_{\hat{h}})$ the Chow class of $\Delta_{\boldsymbol{\tau}_{\curlywedge}}$. Equivalently, we have

$$\mathfrak{spl}_{G,*}[\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\curlywedge})] = [\Delta_{\boldsymbol{\tau}_{\curlywedge}}] \cap [\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty) \times \prod_{V\in\mathbf{V}_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)]. \tag{5.42}$$

Applying (5.42) and the virtual push-forward of [30], we obtain

$$\mathrm{spl}_{G,*}[\mathscr{U}^G(\boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}} = \widetilde{\mathbf{ev}}^*[\Delta_{\boldsymbol{\tau}_{\curlywedge}}] \cap \left([\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_\infty)]^{\mathrm{red}} \times \prod_{V\in\mathbf{V}_0} [\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)]^{\mathrm{vir}} \right). \tag{5.43}$$

On the other hand, by Proposition 5.6 (2) the perfect obstruction theory $\varphi^{\mathrm{red}}_{\mathbf{ev}_G}$ of $\mathbf{ev}_G$ pulls back to the reduced perfect obstruction theory (4.38) of $\mathbf{ev}$. Thus by Proposition (5.8) and the virtual push-forward again, we have

$$\mathrm{spl}_{\log,*}[\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}} = \mu(\boldsymbol{\tau}_{\curlywedge})\cdot[\mathscr{U}^{G}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}}. \tag{5.44}$$

Combining (5.43) and (5.44), we obtain

Lemma 5.9.

$$\mathrm{spl}_{\infty|0,*}[\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\curlywedge})]^{\mathrm{red}} = \mu(\boldsymbol{\tau}_{\curlywedge})\cdot\widetilde{\mathbf{ev}}^{*}[\Delta_{\boldsymbol{\tau}_{\curlywedge}}]\cap\left([\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\infty})]^{\mathrm{red}}\times\prod_{V\in\mathbf{V}_0}[\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_i)]^{\mathrm{vir}}\right)\quad\square$$

5.3. **Decomposing infinity contributions.** To obtain an effective formula, we further break $[\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\infty})]^{\mathrm{red}}$ to contributions from each individual vertex. However, there is no morphism from $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\infty})$ to $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{V})$ for each $V\in\mathbf{V}_{\infty}$ since punctured R-maps parameterized by $\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\infty})$ do not in general admit uniform maximal degeneracies along the component of V. In §6, we will solve this issue by using log alignments §C and log blow-ups to construct morphisms of log stacks

$$\begin{tikzcd}[column sep=small] & \widehat{\mathfrak{U}}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty}) := \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_{\infty})^{a}|\emptyset} \arrow[dl, "(6.16)"'] \arrow[dr] & \\ \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty}) & & \prod_{V\in\mathbf{V}_{\infty}}\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{V}) \end{tikzcd} \tag{5.45}$$

where the right arrow will be given by (6.8) and (6.14).

On one hand, the left arrow of (5.45) leads to a Cartesian diagram

$$\begin{tikzcd} \widehat{\mathscr{U}}(\mathfrak{P},\boldsymbol{\tau}_{\infty}) \arrow[r, "\widehat{F}_{\mathscr{U}}"] \arrow[d, "\widehat{\mathbf{ev}}_{\infty}"'] & \mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\infty}) \arrow[d, "\mathbf{ev}_{\infty}"] \\ \widehat{\mathfrak{U}}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty}) \arrow[r, "\widehat{F}_{\mathfrak{U}}"] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty}) \end{tikzcd}$$

with strict tautological arrows. The moduli interpretation of $\widehat{\mathfrak{U}}^{\mathrm{ev}}(\mathcal{A},\tau_{\infty})$, hence $\widehat{\mathscr{U}}(\mathfrak{P},\boldsymbol{\tau}_{\infty})$ will be given in §6.2.

Remark 5.10. Roughly speaking, the stack $\widehat{\mathscr{U}}(\mathfrak{P},\boldsymbol{\tau}_{\infty})$ has the following properties.

(1) It admits a stable punctured R-map

$$\widehat{f}_{\boldsymbol{\tau}_{\infty}}:\widehat{C^{\circ}_{\boldsymbol{\tau}_{\infty}}} := \sqcup_{V\in\mathbf{V}_{\infty}}\widehat{C^{\circ}_{\boldsymbol{\tau}_{\infty},V}}\to\mathfrak{P}$$

marked by $\boldsymbol{\tau}_{\infty}$ with uniform maximal degeneracies, defining the left arrow in (5.45).

(2) For each vertex V, the restriction $\widehat{f}_{\boldsymbol{\tau}_{\infty}}|_{\widehat{C^{\circ}_{\boldsymbol{\tau}_{\infty},V}}}$ is marked by $\boldsymbol{\tau}_{V}$ with uniform maximal degeneracy $e_{\max,V}\in\overline{\mathcal{M}}_{\widehat{\mathscr{U}}(\mathfrak{P},\boldsymbol{\tau}_{\infty})}$, defining the right arrow in (5.45).

(3) The set of degeneracies $\{e_{\max,V}\}_{V\in\mathbf{V}_{\infty}}$ are geometric fiberwise totally ordered under the natural monoid order.

The stack $\widehat{\mathscr{U}}(\mathfrak{P},\boldsymbol{\tau}_{\infty})$ admits further configurations as in §6.1 in order for proving formulas in Theorem 6.10 and Theorem 6.12 needed in the localization calculation of [17]. We refer to §6.2 for further details.

Pulling back the perfect obstruction theory (5.17) of $\mathbf{ev}_{\infty}$, we obtain the reduced perfect obstruction theory

$$\varphi^{\mathrm{red}}_{\widehat{\mathbf{ev}}_{\infty}}:\mathbb{T}_{\widehat{\mathbf{ev}}_{\infty}}\longrightarrow\mathbb{E}^{\mathrm{red}}_{\widehat{\mathbf{ev}}_{\infty}} := \widehat{F}^{*}_{\mathscr{U}}\mathbb{E}^{\mathrm{red}}_{\mathbf{ev}_{\infty}}.$$

of $\widehat{\mathbf{ev}}_\infty$. In Lemma 6.5, we will verify that $\widehat{\mathfrak{U}}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty)$ is pure-dimensional, and the morphism $\widehat{F}_{\mathfrak{U}}$, hence $\widehat{F}_{\mathscr{U}}$ are proper and birational. These lead to the reduced virtual cycle $[\widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty)]^{\mathrm{red}}$ of $\widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty)$ satisfying the virtual push-forward

$$\widehat{F}_{\mathscr{U},*}[\widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty)]^{\mathrm{red}} = [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\infty)]^{\mathrm{red}}. \tag{5.46}$$

On the other hand, the right arrow of (5.45) leads to a Cartesian diagram

$$\begin{tikzcd}
\widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty) \arrow[r, "\widehat{\mathrm{spl}}_\infty"] \arrow[d] & \prod_{V \in \mathbf{V}_\infty} \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V) \arrow[d, "\prod F_V"] \\
\widehat{\mathfrak{U}}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty) \arrow[r] & \prod_{V \in \mathbf{V}_\infty} \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_V)
\end{tikzcd}$$

We will show that

Lemma 5.11 (Corollary 6.13)**.**

$$\widehat{\mathrm{spl}}_{\infty,*}[\widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty)]^{\mathrm{red}} = (-\tilde{r})^{|V(G_\infty)|-1} \prod_{V \in \mathbf{V}_\infty} [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}}.$$

Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge) \arrow[r, "\mathrm{spl}_{\infty|0}"] \arrow[dr, bend right=10, "\mathrm{spl}_{\boldsymbol{\tau}_\curlywedge}"'] & \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\infty) \times \prod_{V \in \mathbf{V}_0} \mathscr{R}(\boldsymbol{\tau}_V) \arrow[d, "\mathrm{spl}_\infty \times \prod_V id_V"] & \widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty) \times \prod_{V \in \mathbf{V}_0} \mathscr{R}(\boldsymbol{\tau}_V) \arrow[l] \arrow[d, "\widehat{\mathrm{spl}}_\infty \times \prod_V id_V"'] \\
& \prod_{V \in \mathbf{V}(G)} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) & \prod_{V \in \mathbf{V}_\infty} \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V) \times \prod_{V \in \mathbf{V}_0} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \arrow[l, "F_{\mathbf{V}_\infty} \times \prod_V id_V"']
\end{tikzcd} \tag{5.47}$$

where $id_V \colon \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \to \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)$ is the identity, and $F_{\mathbf{V}_\infty}$ is the product of the tautological morphism $\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V) \to \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)$ for $V \in \mathbf{V}_\infty$.

For each $V \in \mathbf{V}(G)$, denote again by

$$\mathrm{ev}_{\mathbf{S}_V} := \prod_{h \in \mathbf{S}_V} \mathrm{ev}_h \colon \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \to \prod_{h \in \mathbf{S}_V} \overline{\gamma}_h$$

the evaluation along half-edges in $\mathbf{S}_V$, and write

$$\mathrm{ev}_{\mathbf{S}} := \prod_{V \in \mathbf{V}(G)} \mathrm{ev}_{\mathbf{S}_V} \colon \prod_{V \in \mathbf{V}(G)} \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V) \longrightarrow \prod_{\{h, \hat{h}\} \in \mathbf{E}(G)} \overline{\gamma}_h \times \overline{\gamma}_{\hat{h}}.$$

We then compute that

$$\begin{aligned}
&\mathrm{spl}_{\boldsymbol{\tau}_\curlywedge,*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\curlywedge)]^{\mathrm{red}} \\
&= \mu(\boldsymbol{\tau}_\curlywedge) \cdot \mathrm{ev}_{\mathbf{S}}^*[\Delta_{\boldsymbol{\tau}_\curlywedge}] \cap \left(\mathrm{spl}_{\infty,*}[\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\infty)]^{\mathrm{red}} \times \prod_{V \in \mathbf{V}_0} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right) \\
&= \mu(\boldsymbol{\tau}_\curlywedge) \cdot \mathrm{ev}_{\mathbf{S}}^*[\Delta_{\boldsymbol{\tau}_\curlywedge}] \cap \left(F_{\mathbf{V}_\infty,*} \widehat{\mathrm{spl}}_{\infty,*}[\widehat{\mathscr{U}}(\mathfrak{P}, \boldsymbol{\tau}_\infty)]^{\mathrm{red}} \times \prod_{V \in \mathbf{V}_0} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right) \\
&= (-\tilde{r})^{|V(G_\infty)|-1} \mu(\boldsymbol{\tau}_\curlywedge) \cdot \mathrm{ev}_{\mathbf{S}}^*[\Delta_{\boldsymbol{\tau}_\curlywedge}] \cap \left(F_{\mathbf{V}_\infty,*} \left(\prod_{V \in \mathbf{V}_\infty} [\mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}} \right) \times \prod_{V \in \mathbf{V}_0} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right) \\
&= (-\tilde{r})^{|V(G_\infty)|-1} \mu(\boldsymbol{\tau}_\curlywedge) \cdot \mathrm{ev}_{\mathbf{S}}^*[\Delta_{\boldsymbol{\tau}_\curlywedge}] \cap \left(\prod_{V \in \mathbf{V}_\infty} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{red}} \times \prod_{V \in \mathbf{V}_0} [\mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V)]^{\mathrm{vir}} \right)
\end{aligned}$$

where the first equality is from Lemma 5.9, and second one follows from the commutativity of Diagram (5.47), the third line is Lemma 5.11, and we recall the notation (5.7) for the last line. To summarize, we have shown that

Theorem 5.12. *For a decorated $\Yup$-type $\boldsymbol{\tau}_{\Yup}$ of rigid tropical curves as in* (5.1)*, we have*

$$\mathrm{spl}_{\boldsymbol{\tau}_{\Yup},*}[\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\Yup})]^{\mathrm{red}} = (-\tilde{r})^{|V(G_\infty)|-1}\mu(\boldsymbol{\tau}_{\Yup})\cdot\mathrm{ev}^*_{\mathbf{S}}[\Delta_{\boldsymbol{\tau}_{\Yup}}]\cap\left(\prod_{V\in\mathbf{V}_\infty}[\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)]^{\mathrm{red}}\times\prod_{V\in\mathbf{V}_0}[\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)]^{\mathrm{vir}}\right).$$

5.4. **Proof of Theorem 5.2.** Consider the commutative diagram

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\Yup}) \arrow[dr, "\mathrm{spl}^G_{\boldsymbol{\tau}_{\Yup}}"] \arrow[drr, bend left, "\mathrm{spl}_{\boldsymbol{\tau}_{\Yup}}"] \arrow[ddr, bend right, "\mathrm{ev}_{\mathbf{E}}"'] & & \\
& \mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\Yup}) \arrow[r] \arrow[d] & \prod_{V\in\mathbf{V}(G)}\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V) \arrow[d, "\mathrm{ev}_{\mathbf{S}}"] \\
& \prod_{x\in\mathbf{E}(G)}\overline{\gamma}_x \arrow[r, "\Delta_{\boldsymbol{\tau}_{\Yup}}"'] & \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)}\overline{\gamma}_h\times\overline{\gamma}_{\hat{h}}
\end{tikzcd} \tag{5.48}$$

where the square is Cartesian. By construction, $\mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\Yup})$ parameterizes stable punctured R-maps marked by $\sqcup_{V\in\mathbf{V}(G)}\boldsymbol{\tau}_V$ with their underlying R-maps glued along $\mathbf{E}(G)$.

We have a commutative diagram

$$\begin{tikzcd}
\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\Yup}) \arrow[rr, "\mathrm{spl}^G_{\boldsymbol{\tau}_{\Yup}}"] \arrow[dr, "F_{\mathscr{M}}"'] & & \mathscr{R}^G(\mathfrak{P},\boldsymbol{\tau}_{\Yup}) \arrow[dl, "F^G_{\mathscr{M}}"] \\
& \mathscr{M}_{g,n}(\mathcal{X},\beta) &
\end{tikzcd} \tag{5.49}$$

By Theorem 5.12 and the commutativity of (5.48), we obtain that

$$\mathrm{spl}^G_{\boldsymbol{\tau}_{\Yup},*}[\mathscr{U}(\mathfrak{P},\boldsymbol{\tau}_{\Yup})]^{\mathrm{red}} = (-\tilde{r})^{|\mathbf{V}_\infty|-1}\mu(\boldsymbol{\tau}_{\Yup})\cdot\Delta^!_{\boldsymbol{\tau}_{\Yup}}\left(\prod_{V\in\mathbf{V}_\infty}[\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)]^{\mathrm{red}}\times\prod_{V\in\mathbf{V}_0}[\mathscr{R}(\mathfrak{P},\boldsymbol{\tau}_V)]^{\mathrm{vir}}\right).$$

Further pushing forward to $\mathscr{M}_{g,n}(\mathcal{X},\beta)$ and applying the commutativity of (5.49), we obtain Theorem 5.2. ☐

5.5. **Proof of Proposition 5.8.**

5.5.1. *A reduction.* We first establish the proper and representability in Proposition 5.8. Observe that (5.37) extends to a commutative diagram

$$\begin{tikzcd}
\mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_{\Yup}) \arrow[r, "\mathfrak{spl}_{\log}"'] \arrow[d, "F_\tau"'] \arrow[rr, bend left, "\mathfrak{spl}_{\infty|0}"] & \mathfrak{U}^{\mathrm{ev},G}(\mathcal{A},\tau_{\Yup}) \arrow[r, "\mathfrak{spl}_G"'] \arrow[d, "F^G_\tau"] & \mathfrak{U}^{\mathrm{ev}}(\mathcal{A},\tau_\infty)\times\prod_{V\in\mathbf{V}_0}\mathfrak{M}^{\mathrm{ev}}(\tau_V) \arrow[d, "F_{\mathbf{V}}:=\prod_{V\in\mathbf{V}(G)}F_{\tau_{\mathbf{V}}}"] \\
\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau) \arrow[r, "\overline{\mathfrak{spl}}_{\log}"'] & \mathfrak{M}^{\mathrm{ev},G}(\mathcal{A},\tau) \arrow[r, "\overline{\mathfrak{spl}}_G"'] \arrow[d, "\overline{\mathrm{ev}}^G_{\mathbf{E}}"'] & \prod_{V\in\mathbf{V}_\infty}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)\times\prod_{V\in\mathbf{V}_0}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \arrow[d, "\overline{\mathrm{ev}}_{\mathbf{S}}"] \\
& \prod_{x\in\mathbf{E}(G)}\overline{\gamma}_x \arrow[r, "\Delta_{\boldsymbol{\tau}_{\Yup}}"'] & \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)}\overline{\gamma}_h\times\overline{\gamma}_{\hat{h}}
\end{tikzcd} \tag{5.50}$$

where the three top vertical arrows are the tautological ones given by (3.12), and the two squares on the right are both Cartesian. By Proposition 3.13 both F_τ and $F_{\mathbf{V}}$ are proper and representable. Hence F^G_τ is also proper and representable. Therefore, we the properness and representability of $\mathfrak{spl}_{\log}$ follows immediately from the following.

Proposition 5.13. *$\overline{\mathfrak{spl}}_{\log}$ is representable and proper.*

The proof of this proposition will be postponed to the end of §5.5.2 after the discussion on gluing of puncture maps below.

5.5.2. *Gluing of punctured maps.* Following [3, §5.2, §5.3], we discuss the gluing of punctured maps in the setting of this paper. A similar discussion but in a different setting can be found in [23].

For each $V \in \mathbf{V}(G)$, denote by $\mathfrak{M}'(\mathcal{A}, \tau_V)$ the moduli of punctured maps to $\mathcal{A}$ weakly marked by τ_V as in §B.3.4. Using Lemma 2.6 and replacing markings by weak markings in (2.53), we define the log evaluation stack $\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)$ with its universal family

$$\begin{tikzcd}
\bigsqcup_{x\in\mathbf{L}_V\cup\mathbf{S}_V} \mathfrak{p}_{x,\tau_V} \arrow[r, "ev_{\tau_V}"] \arrow[d] & \bigsqcup_{x\in\mathbf{L}_V\cup\mathbf{S}_V} \boldsymbol{\gamma}_x \arrow[d] \\
\mathfrak{C}^\circ_{\tau_V} \arrow[r, "\mathfrak{f}_{\tau_V}"] & \mathcal{A}
\end{tikzcd} \tag{5.51}$$

where $\mathfrak{f}_{\tau_V}$ is weakly marked by τ_V, and $\mathfrak{p}_{x,\tau_V} \to \boldsymbol{\gamma}_x$ is defined by the sector $\overline{\boldsymbol{\gamma}}_x$ given by the sector decoration of $\boldsymbol{\tau}_{\curlywedge}$.

Remark 5.14. Note that the tautological morphism $\mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V) \to \mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)$ given by (B.35) is an isomorphism by Remark B.21. While for each vertex V we are mainly interested in punctured maps marked by τ_V, Proposition 5.15 below suggests that gluing of punctured maps occur naturally in the weakly marked setting. We refer to [3, §5.2, §5.3] for the throughout discussions on this point.

Consider the fiber production over $\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)$ in the fs category

$$\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)} := \prod_{x\in\mathbf{S}_V} (\mathfrak{p}_{x,\tau_V})^{\mathrm{S}},$$

where $(\mathfrak{p}_{x,\tau_V})^{\mathrm{S}} \to (\mathfrak{p}_{x,\tau_V})$ is the saturation morphism. The calculation in [3, Proposition 5.7] implies that the morphism

$$\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)} \to \prod_{x\in\mathbf{S}_V} \underline{\mathfrak{p}_{x,\tau_V}}, \tag{5.52}$$

where the fiber product on the right is taken over $\underline{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)}$ in the category of usual schemes, induces an isomorphism on the reduction of the underlying stacks.

Pulling back (5.51), we obtain the universal diagram over $\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)}$

$$\begin{tikzcd}
\bigsqcup_{x\in\mathbf{L}_V\cup\mathbf{S}_V} \widetilde{\mathfrak{p}_{x,\tau_V}} \arrow[r, "\widetilde{ev_{\tau_V}}"] \arrow[d] & \bigsqcup_{x\in\mathbf{L}_V\cup\mathbf{S}_V} \boldsymbol{\gamma}_x \arrow[d] \\
\widetilde{\mathfrak{C}^\circ_{\tau_V}} \arrow[r, "\widetilde{\mathfrak{f}_{\tau_V}}"] & \mathcal{A}
\end{tikzcd} \tag{5.53}$$

together with a section $s_x \colon \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\tau_V)} \to \widetilde{\mathfrak{p}_{x,\tau_V}}$ for each $x \in \mathbf{S}_V$. Further composing s_x with $\widetilde{ev_{\tau_V}}$, we obtain the evaluation $\widetilde{ev}_x \colon \widetilde{\mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V)} \to \boldsymbol{\gamma}_x$. Taking products, we write

$$\widetilde{ev}_{\mathbf{S}} := \prod_{x\in\mathbf{S}_\infty\sqcup\mathbf{S}_0} \widetilde{ev}_x \colon \prod_{V\in\mathbf{V}_\infty} \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)} \times \prod_{V\in\mathbf{V}_0} \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A}, \tau_V)} \to \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)} \boldsymbol{\gamma}_h \times \boldsymbol{\gamma}_{\hat{h}}$$

On the other hand, consider the stack of weakly marked punctured maps $\mathfrak{M}'(\mathcal{A},\tau)$ as in §B.3.4. Using Lemma 2.6, Remark 2.12 and replacing markings by weak markings in (2.53) with $\mathbf{S} = \mathbf{E}(G)$, we define the log evaluation stack $\mathfrak{M}'^{,\mathrm{ev}}(\tau)$ with its universal family

$$\begin{tikzcd}
\bigsqcup_{x\in\mathbf{L}(G)\cup\mathbf{E}(G)} \mathfrak{p}_{x,\tau} \arrow[r, "\sqcup ev_{x,\tau}"] \arrow[d] & \bigsqcup_{x\in\mathbf{L}(G)\cup\mathbf{E}(G)} \gamma_x \arrow[d] \\
\mathfrak{C}^{\circ}_{\tau} \arrow[r, "\mathfrak{f}_\tau"] & \mathcal{A}
\end{tikzcd} \tag{5.54}$$

where $\mathfrak{f}_\tau$ is weekly marked by τ, and $\mathfrak{P}_{x,\tau} \to \gamma_x$ is defined by $\overline{\gamma}_x$ given by the sector decoration of $\boldsymbol{\tau}_{\curlywedge}$. We follow the same notations as in (5.8) where for each $x = \{h,\hat{h}\} \in \mathbf{E}(G)$ with $h \in \mathbf{S}_\infty$ we choose $\mathfrak{p}_{x,\tau}$ be the gerbe corresponding to h.

Consider the fiber production over $\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)$ in the fs category

$$\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} := \prod_{x\in\mathbf{E}(G)} \mathfrak{p}_{x,\tau}.$$

The calculation in [3, Proposition 5.7] implies that the morphism

$$\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \to \prod_{x\in\mathbf{E}(G)} \underline{\mathfrak{p}_{x,\tau}}, \tag{5.55}$$

where the fiber product on the right is taken over $\underline{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)}$ in the category of usual schemes, is an isomorphism of the underlying stacks.

Pulling back (5.54), we obtain the universal diagram over $\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)}$

$$\begin{tikzcd}
\bigsqcup_{x\in\mathbf{L}(G)\cup\mathbf{E}(G)} \widetilde{\mathfrak{p}_{x,\tau}} \arrow[r, "\widetilde{ev_\tau}"] \arrow[d] & \bigsqcup_{x\in\mathbf{L}(G)\cup\mathbf{S}(G)} \gamma_x \arrow[d] \\
\widetilde{\mathfrak{C}^{\circ}_{\tau}} \arrow[r, "\widetilde{\mathfrak{f}_\tau}"] & \mathcal{A}
\end{tikzcd}$$

together with a section $s_x \colon \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \to \widetilde{\mathfrak{p}_{x,\tau}}$ for each $x \in \mathbf{E}(G)$. We again obtain the evaluation $\widetilde{ev_x}\colon \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \to \gamma_x$ induced by $\widetilde{ev_\tau}$. Denote by

$$\widetilde{ev_{\mathbf{E}}} := \prod \widetilde{ev_x}\colon \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \to \prod_{x\in\mathbf{E}(G)} \gamma_x.$$

Taking partial normalization along nodes of $\mathbf{E}(G)$ and repeating the discussions as in Proposition 5.5, we obtain a canonical morphism

$$\widetilde{\mathfrak{spl}}' \colon \widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \longrightarrow \prod_{V\in\mathbf{V}_\infty} \mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V) \times \prod_{V\in\mathbf{V}_0} \mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V).$$

For each $x = \{h,\hat{h}\} \in \mathbf{E}(G)$ with $h \in \mathbf{S}_\infty$, consider the involuted diagonal of sectors

$$\widetilde{\Delta_x} := id \times \iota_\gamma \colon \gamma_x \to \gamma_h \times \gamma_{\hat{h}}$$

where ι_γ is the involution as in (2.11). Taking fiber product, we have

$$\widetilde{\Delta_{\boldsymbol{\tau}_{\curlywedge}}} := \prod_{x\in\mathbf{E}(G)} \gamma_x \to \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)} \gamma_h \times \gamma_{\hat{h}}.$$

This is a finite and representable morphism. The following is an analogue of [3, Corollary 5.15].

Proposition 5.15. *There is a canonical Cartesian diagram in the fs category*

$$\begin{tikzcd}
\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \arrow[r, "\widetilde{\mathfrak{spl}}'"] \arrow[d, "\widetilde{ev}_{\mathbf{E}}"'] & \prod_{V\in\mathbf{V}_\infty}\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)}\times\prod_{V\in\mathbf{V}_0}\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)} \arrow[d, "\widetilde{ev}_{\mathbf{S}}"] \\
\prod_{x\in\mathbf{E}(G)}\gamma_x \arrow[r, "\widetilde{\Delta}_{\mathbf{E}}"'] & \prod_{\{h,\hat{h}\}\in\mathbf{E}(G)}\gamma_h\times\gamma_{\hat{h}}
\end{tikzcd}$$

In particular, $\widetilde{\mathfrak{spl}}'$ *is finite and representable.*

Proof. This follows from a similar proof as in [3, Theorem 5.8, Corollary 5.15] but with orbifold structures and nodal involutions along evaluations. We recall the proof below and refer to [3] for additional details.

Consider a morphism from an fs log scheme W to the fiber product in the fs category

$$W\to\left(\prod_{x\in\mathbf{E}(G)}\gamma_x\right)\times_{\prod_{\{h,\hat{h}\}\in\mathbf{E}(G)}\gamma_h\times\gamma_{\hat{h}}}\left(\prod_{V\in\mathbf{V}_\infty}\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)}\times\prod_{V\in\mathbf{V}_0}\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)}\right). \tag{5.56}$$

The goal is to show that this morphism factors through $\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\tau)}$ canonically.

For each V, pulling back (5.53) we obtain over W a commutative diagram

$$\begin{tikzcd}
\coprod_{x\in\mathbf{L}_V\cup\mathbf{S}_V}\widetilde{\mathfrak{p}_{x,W}} \arrow[r, "\widetilde{ev_{\tau_V,W}}"] \arrow[d] & \coprod_{x\in\mathbf{L}_V\cup\mathbf{S}_V}\gamma_x \arrow[d] \\
\widetilde{\mathfrak{C}^\circ_{\tau_V,W}} \arrow[r, "\widetilde{\mathfrak{f}_{V,W}}"] & \mathcal{A}
\end{tikzcd} \tag{5.57}$$

together with a section $s_{x,W}\colon W\to\widetilde{\mathfrak{p}_{x,\tau_V}}$ for each $x\in\mathbf{S}_V$. These data further satisfy the compatibilities that for any edge $x=\{h,\hat{h}\}\in\mathbf{E}(G)$ with $h\in\mathbf{S}_\infty$ the following diagrams on the logarithmic level and on the underlying level are commutative respectively

$$\begin{tikzcd}
W \arrow[r, "s_{h,W}"] \arrow[dr, bend right, "s_{\hat{h},W}"'] & \mathfrak{p}_{h,W} \arrow[r] & \gamma_h \arrow[r] \arrow[d, "\iota_\gamma"] & \mathfrak{P}_{\mathbf{k}} \arrow[d, "\iota_\omega"] \\
& \mathfrak{p}_{\hat{h},W} \arrow[r] & \gamma_{\hat{h}} \arrow[r] & \mathfrak{P}_{\mathbf{k}}
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\underline{\mathfrak{p}_{h,W}} \arrow[r] \arrow[d, "\underline{\iota}"] & \underline{\gamma_h} \arrow[d, "\underline{\iota_\gamma}"] \\
\underline{\mathfrak{p}_{\hat{h},W}} \arrow[r] & \underline{\gamma_{\hat{h}}}
\end{tikzcd} \tag{5.58}$$

where $\underline{\iota}$ is the involution inverting the band.

By the commutativity of the underlying diagram, the underlying of (5.57) glues to

$$\begin{tikzcd}
\coprod_{x\in\mathbf{L}(G)\cup\mathbf{E}(G)}\underline{\widetilde{\mathfrak{p}_{x,W}}} \arrow[r, "\underline{\widetilde{ev_{\tau,W}}}"] \arrow[d] & \coprod_{x\in\mathbf{L}(G)\cup\mathbf{S}(G)}\underline{\gamma_x} \arrow[d] \\
\underline{\widetilde{\mathfrak{C}^\circ_{\tau,W}}} \arrow[r, "\underline{\widetilde{\mathfrak{f}_{\tau,W}}}"] & \underline{\mathcal{A}}
\end{tikzcd} \tag{5.59}$$

together with underlying sections $\underline{s_{h,W}}\colon\underline{W}\to\underline{\widetilde{\mathfrak{p}_{x,W}}}:=\underline{\widetilde{\mathfrak{p}_{h,W}}}$ for each $x=\{h,\hat{h}\}\in\mathbf{E}(G)$. Here $\underline{\widetilde{\mathfrak{C}^\circ_{\tau,W}}}\to\underline{W}$ is a pre-stable curve obtained by gluing $\underline{\widetilde{\mathfrak{C}^\circ_{\tau_V,W}}}$ via the identification $\underline{\mathfrak{p}_{h,W}}\xrightarrow{\underline{\iota}}\underline{\mathfrak{p}_{\hat{h},W}}$. To show that (5.56) factors through $\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\tau)}$ canonically, we show that (5.59) and $\underline{s_{h,W}}$ can be lift to the logarithmic level determined uniquely by (5.57) and $s_{x,W}$.

First we install log structures on the over curve $\underline{\widetilde{\mathfrak{C}^\circ_{\tau,W}}}^\circ := \underline{\widetilde{\mathfrak{C}^\circ_{\tau,W}}} \setminus \sqcup_{x\in\mathbf{E}(G)}\underline{\widetilde{\mathfrak{p}_{x,W}}}$ by pulling back log structures from $\widetilde{\mathfrak{C}^\circ_{\tau_V,W}}$. This leads to a unique log map $\widetilde{\mathfrak{f}_{\tau,W}}^\circ : \widetilde{\mathfrak{C}^\circ_{\tau,W}}^\circ \to \mathcal{A}$ defined on the open curve induced by $\widetilde{\mathfrak{f}_{V,W}}$ for all V. It suffices to extend $\widetilde{\mathfrak{f}_{\tau,W}}^\circ$ across all nodal sections.

As in [3, Proof of Thm 5.8, Step 2], the sections $s_{h,W}, s_{\hat{h},W}$ for $x = \{h, \hat{h}\}$ on the logarithmic level further imply that

(1) $\widetilde{\mathfrak{C}^\circ_{\tau,W}}^\circ$ extends uniquely to a punctured curve $\widetilde{\mathfrak{C}^\circ_{\tau,W}} \to W$ across all nodal sections.
(2) For each $x = \{h, \hat{h}\}$, let $\widehat{\mathfrak{p}_{h,W}}$ and $\widehat{\mathfrak{p}_{\hat{h},W}}$ be the two gerbes of $\widetilde{\mathfrak{C}^\circ_{\tau,W}}$ corresponding to h and $\hat{h}$ respectively. Then we obtain sections $\widehat{s_{h,W}}\colon W \to \widehat{\mathfrak{p}_{h,W}}$ and $\widehat{s_{\hat{h},W}}\colon W \to \widehat{\mathfrak{p}_{\hat{h},W}}$ lifting $\underline{s_{h,W}}$ and $\underline{s_{\hat{h},W}}$ respectively.

Furthermore, these lifts on the logarithmic level have the following compatibilities

$$\mathcal{M}^{gp}_{\widehat{\mathfrak{p}_{h,W}}} = \mathcal{M}^{gp}_{\mathfrak{p}_{h,W}}, \qquad \mathcal{M}^{gp}_{\widehat{\mathfrak{p}_{\hat{h},W}}} = \mathcal{M}^{gp}_{\mathfrak{p}_{\hat{h},W}} \tag{5.60}$$

and

$$(\widehat{s_{h,W}}^{\flat})^{gp} = (s^{\flat}_{h,W})^{gp}, \qquad (\widehat{s_{\hat{h},W}}^{\flat})^{gp} = (s^{\flat}_{\hat{h},W})^{gp}. \tag{5.61}$$

Indeed, these logarithmic lifts $\widetilde{\mathfrak{C}^\circ_{\tau,W}}^\circ, \widehat{s_{h,W}}, \widehat{s_{\hat{h},W}}$ are constructed first étale locally along nodal sections by applying [3, Lemma 5.10], then glued together thanks to the uniqueness of [3, Remark 5.11]. The étale local construction can be applied in our setting with orbifold domain curves as the orbifold isotropy groups act trivially on the characteristic monoids involved in the construction of [3, Lemma 5.10].

Next we extend $\widetilde{\mathfrak{f}_{\tau,W}}^\circ$ across all nodal sections. By the strictness of ι_ω, the left diagram in (5.58) extends to a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "s_{h,W}"] \arrow[dr, bend right=15, "s_{\hat{h},W}"'] & \mathfrak{p}_{h,W} \arrow[r] & \gamma_h \arrow[r] \arrow[d, "\iota_\gamma"] & \mathfrak{P}_{\mathbf{k}} \arrow[d, "\iota_\omega"] \arrow[dr, bend left=15] & \\
& \mathfrak{p}_{\hat{h},W} \arrow[r] & \gamma_{\hat{h}} \arrow[r] & \mathfrak{P}_{\mathbf{k}} \arrow[r] & \mathcal{A}
\end{tikzcd}$$

where both arrows to $\mathcal{A}$ are strict. The compatibilities (5.60) and (5.61) hence imply a commutative diagram

$$\begin{tikzcd}
W \arrow[r, "\widehat{s_{h,W}}"] \arrow[dr, bend right=15, "\widehat{s_{\hat{h},W}}"'] & \widehat{\mathfrak{p}_{h,W}} \arrow[r] \arrow[d, "\iota"] & \gamma_h \arrow[r] \arrow[d, "\iota_\gamma"] & \mathfrak{P}_{\mathbf{k}} \arrow[d, "\iota_\omega"] \arrow[dr, bend left=15] & \\
& \widehat{\mathfrak{p}_{\hat{h},W}} \arrow[r] & \gamma_{\hat{h}} \arrow[r] & \mathfrak{P}_{\mathbf{k}} \arrow[r] & \mathcal{A}
\end{tikzcd} \tag{5.62}$$

where ι is inverting the band. As in [3, Proof of Thm 5.8, Step 4], the above commutative diagrams imply that the tropicalizations of $\widetilde{\mathfrak{f}_{\nu(h),W}}$ and $\widetilde{\mathfrak{f}_{\nu(\hat{h}),W}}$ can be glued along the edge $\{h, \hat{h}\}$ as follows.

Indeed, as explained in [3, Remark 5.12] the tropicalizations of $s_{h,W}$ and $s_{\hat{s},W}$ give sections of the corresponding tropical curves

$$\Sigma(s_{h,W})\colon \Sigma(W) \to \Sigma(\widetilde{\mathfrak{C}^\circ_{\nu(h),W}}), \qquad \Sigma(s_{\hat{h},W})\colon \Sigma(W) \to \Sigma(\widetilde{\mathfrak{C}^\circ_{\nu(\hat{h}),W}})$$

Then we may glue $\Sigma(\widetilde{\mathfrak{C}^\circ_{\nu(h),W}})$ and $\Sigma(\widetilde{\mathfrak{C}^\circ_{\nu(\hat{h}),W}})$ by identifying the two tropical sections for all $\{h, \hat{h}\}$, and obtain $\Sigma(\widetilde{\mathfrak{C}^\circ_{\tau,W}})$. Similarly, the tropicalizations of $\widehat{s_{h,W}}$ and $\widehat{s_{\hat{h},W}}$ lead to two sections of the tropical curves

$$\Sigma(\widehat{s_{h,W}}), \Sigma(\widehat{s_{\hat{h},W}})\colon \Sigma(W) \to \Sigma(\widetilde{\mathfrak{C}^{\circ}_{\tau,W}}).$$

Equation (5.61) implies that these two sections of tropical curves agree with $\Sigma(s_{h,W})$ and $\Sigma(s_{\hat{h},W})$, hence is indeed the same section $\Sigma(\widehat{s_{h,\hat{h},W}}) \subset \Sigma(\widetilde{\mathfrak{C}^{\circ}_{\tau,W}})$. Taking tropicalization of the commutative diagram (5.62), we observe that the tropical maps

$$\Sigma(\widetilde{\mathfrak{f}_{\nu_h,W}})\colon \Sigma(\widetilde{\mathfrak{C}^{\circ}_{\nu(h),W}}) \to \Sigma(\mathcal{A}), \qquad \Sigma(\widetilde{\mathfrak{f}_{\nu_{\hat{h}},W}})\colon \Sigma(\widetilde{\mathfrak{C}^{\circ}_{\nu(\hat{h}),W}}) \to \Sigma(\mathcal{A})$$

agree along $\Sigma(\widehat{s_{h,\hat{h},W}})$. Thus we may glue $\Sigma(\widetilde{\mathfrak{f}_{\nu_h,W}})$ and $\Sigma(\widetilde{\mathfrak{f}_{\nu_{\hat{h}},W}})$ along $\Sigma(\widehat{s_{h,\hat{h},W}})$ for all $\{h,\hat{h}\}$ to obtain the glued tropical map $\Sigma(\widetilde{\mathfrak{f}_{\tau,W}})\colon \Sigma(\widetilde{\mathfrak{C}^{\circ}_{\tau,W}}) \to \Sigma(\mathcal{A})$.

By [2, Proposition 2.10], the tropical map $\Sigma(\widetilde{\mathfrak{f}_{\tau,W}})$ implies a unique punctured map $\widetilde{\mathfrak{f}_{\tau,W}}$ extending $\widetilde{\mathfrak{f}_{\tau,W}}^{\circ}$. This completes the proof. □

Consider the following commutative diagram

$$\begin{tikzcd}
\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau) \arrow[r, "\mathfrak{spl}=\overline{\mathfrak{spl}}_G\circ\overline{\mathfrak{spl}}_{\log}"] \arrow[d] & \prod_{V\in\mathbf{V}_\infty}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V)\times\prod_{V\in\mathbf{V}_0}\mathfrak{M}^{\mathrm{ev}}(\mathcal{A},\tau_V) \arrow[d] \\
\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau) \arrow[r, "\mathfrak{spl}'"'] & \prod_{V\in\mathbf{V}_\infty}\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)\times\prod_{V\in\mathbf{V}_0}\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)
\end{tikzcd} \tag{5.63}$$

where the vertical arrows are given by (B.35).

Corollary 5.16. *Both morphisms $\mathfrak{spl}$ and $\mathfrak{spl}'$ are finite and representable.*

Proof. Since the vertical arrows in (5.63) are strict closed embeddings by Proposition B.27, it suffice to prove $\mathfrak{spl}'$ is finite and representable. Consider the following commutative diagram of the underlying stacks

$$\begin{tikzcd}
\underline{\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)}} \arrow[r, "\widetilde{\mathfrak{spl}}'"] \arrow[d, "(5.55)"'] & \prod_{V\in\mathbf{V}(G)}\underline{\widetilde{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)}} \arrow[d, "(5.52)"] \\
\prod_{x\in\mathbf{E}(G)}\underline{\mathfrak{p}_{x,\tau}} \arrow[d] & \prod_{V\in\mathbf{V}(G)}\prod_{x\in\mathbf{S}_V}\underline{\mathfrak{p}_{x,\tau_V}} \arrow[d] \\
\underline{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau)} \arrow[r, "\mathfrak{spl}'"'] & \prod_{V\in\mathbf{V}(G)}\underline{\mathfrak{M}'^{,\mathrm{ev}}(\mathcal{A},\tau_V)}
\end{tikzcd}$$

Since (5.55) is an isomorphism of the underlying stacks, the composition on the left vertical side is just rigidification of nodal gerbes. On the right vertical side, since (5.52) is isomorphic on the reduction, we may replace all stacks in the above diagram by their reductions. Hence the composition on the right vertical side is again rigidification of gerbes from splitting half-edges. The statement follows from the fact that $\widetilde{\mathfrak{spl}}'$ is finite and representable by Proposition 5.15. □

Proof of Proposition 5.13. Noting the $\Delta_{\mathbf{E}}$ is finite and representable, the Cartesian squares in (5.50) imply that $\overline{\mathfrak{spl}}_{\log}$ is also finite and representable. Further using the finiteness of $\mathfrak{spl} = \overline{\mathfrak{spl}}_G \circ \overline{\mathfrak{spl}}_{\log}$ from Corollary 5.16 we conclude that $\overline{\mathfrak{spl}}_G$ is again finite and representable. □

5.5.3. *The degree computation.* It remains to show that $\mathfrak{spl}_{\log}$ is generically finite of degree $\mu(\boldsymbol{\tau}_\curlywedge)$ as in (5.12). By Remark 3.19, there is an open and dense substack $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty)^\circ \subset \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty)$ parameterizing punctured maps with tropical type τ_∞. Similarly, for each $V \in \mathbf{V}_0$ there is an open and dense substack $\mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V)^\circ \subset \mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_V)$ parameterizing non-degenerate log maps. Note again that in (5.37), $\mathrm{ev}_{\mathbf{S}}$ hence $\mathrm{ev}^G_{\mathbf{E}}$ are both fiberations. Thus we obtain an open and dense substack of $\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)$

$$\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)^\circ := \mathfrak{spl}_G^{-1}\left(\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\infty)^\circ \times \prod_{V \in \mathbf{V}_0} \mathfrak{M}^{\mathrm{ev}}(\mathcal{A}, \tau_i)^\circ\right)$$

over which each connected component of the domain curves is irreducible.

Similarly by Remark 3.19, we have an open and dense substack $\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\curlywedge)^\circ \subset \mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\curlywedge)$ parameterizing punctured maps of $\curlywedge$-type $\tau_\curlywedge$. By construction observe that

$$\mathfrak{U}^{\mathrm{ev}}(\mathcal{A}, \tau_\curlywedge)^\circ = \mathfrak{spl}_{\log}^{-1}\left(\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)^\circ\right).$$

Consider any strict geometric point $S \to \mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)^\circ$. Pulling back universal families (5.38) over $\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)^\circ$ along $S \to \mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)^\circ$, we obtain families over S

$$\begin{tikzcd}
\sqcup_{x \in \mathbf{H}(G_\infty)} p_x \arrow[r] \arrow[d] & \sqcup_{x \in \mathbf{H}(G_\infty)} \curlyvee_{\infty,x} \arrow[d] \\
C^\circ_\infty \arrow[r, "\mathfrak{f}^G_{\infty,S}"] & \mathcal{A},
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\sqcup_{x \in \mathbf{H}(G_V)} p_x \arrow[r] \arrow[d] & \sqcup_{x \in \mathbf{H}(G_V)} \curlyvee_{\infty,x} \arrow[d] \\
C^\circ_V \arrow[r, "\mathfrak{f}^G_{V,S}"] & \mathcal{A}
\end{tikzcd}$$

where V runs through vertices in $\mathbf{V}_0$, and the underlying morphisms glue to

$$\begin{tikzcd}
\sqcup_{x \in \mathbf{L}(G) \sqcup \mathbf{E}(G)} p_x \arrow[r] \arrow[d] & \sqcup_{x \in \mathbf{L}(G) \sqcup \mathbf{E}(G)} \overline{\curlyvee}_x \arrow[d] \\
\underline{C^\circ_G} \arrow[r, "\underline{\mathfrak{f}_S}"] & \underline{\mathcal{A}}
\end{tikzcd}$$

We will show that the pre-image

$$W = \mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge) \times_{\mathfrak{U}^{\mathrm{ev},G}(\mathcal{A}, \tau_\curlywedge)^\circ} S$$

consists of $\mu(\boldsymbol{\tau}_\curlywedge)$ many reduced points, hence finishing the proof of Proposition 5.8. The proof is divided into several steps.

Step 1. Choosing local sections over S. First observe that $\overline{\mathcal{M}}_S \cong \mathbb{N}$ with its generator the degeneracy $e_{\max,S}$ of components given by vertices in $\mathbf{V}_\infty$. We fix $\tilde{e}_{\max,S} \in \mathcal{M}_S$ lifting $e_{\max,S}$. For each $x = \{h, \hat{h}\} \in \mathbf{E}(G)$ with $h \in \mathbf{S}_\infty$ and $\hat{h} \in \mathbf{S}_0$, we fix a local section δ_x of $\underline{\mathfrak{f}_S}^* \mathcal{M}_{\mathcal{A}}$ defined in a neighborhood of the node $p_x \subset \underline{C^\circ_G}$. Choosing appropriate local section s_h of $\mathcal{M}_{C^\circ_{\nu(h)}}$ (resp. $s_{\hat{h}}$ of $\mathcal{M}_{C^\circ_{\nu(\hat{h})}}$) defined in an étale neighborhood of the gerbe $p_h \subset C^\circ_{\nu(h)}$ (resp. $p_{\hat{h}} \subset C^\circ_{\nu(\hat{h})}$) such that along p_x we have

$$(\mathfrak{f}^G_{\infty,S})^\flat(\delta_x) = \tilde{e}_{\max} + \mathbf{c}(h) \cdot s_h \qquad (\text{resp. } (\mathfrak{f}^G_{V,S})^\flat(\delta_x) = \mathbf{c}(\hat{h}) \cdot s_{\hat{h}}\ ). \tag{5.64}$$

Step 2. Choosing local sections over W. For any strict geometric point $T \to W$, observe on the characteristic level that $\overline{\mathcal{M}}_T = \sigma^\vee_{\tau_\curlywedge,\mathbb{Z}} \cong \mathbb{N}$. Let $e \in \sigma^\vee_{\tau_\curlywedge,\mathbb{Z}}$ be the generator, and denote by $l = \operatorname{lcm}(\mathbf{c}(x) \mid x \in \mathbf{E}(G))$. By (3.6), we have the maximal degeneracy $e_{\max,W} = l \cdot e \in \sigma^\vee_{\tau_\curlywedge,\mathbb{Z}}$.

Consider a connected strict étale neighborhood $U \to W$ of T such that e lifts to a section $\tilde{e}_U$ of $\mathcal{M}_U$ on U. Let $\mathfrak{f}_U \colon C^\circ_U \to \mathcal{A}$ be the punctured map over U. Since $\mathfrak{f}_U$ is marked by τ, we have the image $\alpha(\tilde{e}_U) = 0$ in $\mathcal{O}_U$. By choosing $\tilde{e}_U$ appropriately, we may assume that the morphism $U \to S$ on the log structure level satisfy

$$\mathfrak{spl}^\flat_{\log}(\tilde{e}_{\max,S}) = l \cdot \tilde{e}_U. \tag{5.65}$$

Step 3. Gluing parameters of the domain punctured curves. The underlying curve $\underline{C}^\circ_G \to \underline{S}$ admits a canonical log structures, denoted by $C^\sharp_G \to S^\sharp$, see §B.1.8. Note that $\overline{\mathcal{M}}_{S^\sharp} \cong \mathbb{N}^{|\mathbf{E}(G)|}$ with a set of generators $\{\ell(x)\}_{x\in\mathbf{E}(G)}$ given by the edge length parameters. By (B.3), for $x = \{h, \hat{h}\}$ as above we have a lift of $\ell(x)$

$$\tilde{\ell}(x) = s_h + s_{\hat{h}} \in \mathcal{M}_{S^\sharp}. \tag{5.66}$$

Note that the underlying map $\underline{\mathfrak{f}_U}$ is obtained by pulling back $\underline{\mathfrak{f}_S}$. Thus the corresponding log curve $C^\circ_U \to C_U$ is obtained by the pull-back of $C^\sharp_G \to S^\sharp$ via a canonical arrow $U \to S^\sharp$. On the characteristic level, we compute that

$$\overline{\mathcal{M}}_{S^\sharp} \to \overline{\mathcal{M}}_U, \qquad \ell(x) \mapsto \frac{l}{\mathbf{c}(x)} e.$$

Hence on the log structures level, we have

$$\mathcal{M}_{S^\sharp} \to \mathcal{M}_U, \qquad \tilde{\ell}(x) \mapsto \frac{l}{\mathbf{c}(x)} \tilde{e}_U + u_x \tag{5.67}$$

for a unique invertible section $u_x \in \mathcal{O}^\times_U$.

Conversly, note that C°_G is obtained by gluing the domain punctured curves over S along edges in $\mathbf{E}(G)$. Thus the choices $\{u_x\}_{x\in\mathbf{E}(G)}$ uniquely determines $C^\sharp_G$. Thus, we may view $\{u_x\}_{x\in\mathbf{E}(G)}$ as the collection of parameters for gluing domain curves.

Step 4. Constraints of gluing parameters. For an edge $x = \{h, \hat{h}\}$ as above, let $p_{x,U} \subset C^\circ_U$ be the corresponding node, and $p_{h,U}, p_{\hat{h},U} \to C^\circ_U$ be the corresponding gerbes. Then we have

$$(\mathfrak{f}^G_{\infty,S})^\flat(\delta_x)|_{p_{h,U}} = \mathfrak{f}_U^\flat(\delta_x)|_{p_{h,U}} = \mathfrak{f}_U^\flat(\delta_x)|_{p_{\hat{h},U}} = (\mathfrak{f}^G_{V,S})^\flat(\delta_x)|_{p_{\hat{h},U}} \tag{5.68}$$

where the two equalities on the two ends follow from that $\mathfrak{f}_U|_{p_{x,U}}$ is obtained by gluing $\mathfrak{f}^G_{\infty,S}|_{p_h}$ and $\mathfrak{f}^G_{\nu(\hat{h}),S}|_{p_{\hat{h}}}$ along x. Since the punctured map $\mathfrak{f}_U$ is determined by gluing $\mathfrak{f}^G_{\infty,S}$ and $\mathfrak{f}^G_{V,S}$ for all $V \in \mathbf{V}_0$ along edges as above, we see that the gluing parameters $\{u_x\}_{x\in\mathbf{E}(G)}$ determines not only the domain curve, but also the punctured map $\mathfrak{f}_U$ over U.

Combining the compatibility (5.68) along edges with (5.64), we obtain that

$$\tilde{e}_{\max} + \mathbf{c}(h) \cdot s_h = \mathbf{c}(\hat{h}) \cdot s_{\hat{h}}, \qquad \text{in } \mathcal{M}_{C^\circ_U}|_{p_{x,U}}$$

Further applying (5.66), (5.67) and $\mathbf{c}(x) = \mathbf{c}(\hat{h}) = -\mathbf{c}(h)$, we have

$$\tilde{e}_{\max} = \mathbf{c}(x) \cdot \tilde{\ell}(x) = l \cdot \tilde{e} + u_x^{\mathbf{c}(x)}, \qquad \text{in } \mathcal{M}_U \tag{5.69}$$

Comparing with (5.65), we conclude that

$$u_x^{\mathbf{c}(x)} = 1 \qquad \text{in } \mathcal{O}^\times_U. \tag{5.70}$$

This is a set of further constraints on $\{u_x\}_{x\in\mathbf{E}(G)}$ obtained from gluing maps.

Thus $u_x \in \mathbf{k}^\times$ is a $\mathbf{c}(x)$-th root of unit for each $x \in \mathbf{E}(G)$. The connectedness of U implies that it is a reduced point, hence W is a collection of reduced points.

Step 5. Counting fibers over S. Reversing the discussions in **Step 3** and **Step 4**, we observe that fixing $\tilde{e}_{\max,S} \in \mathcal{M}_S$, each points $U \in W$ together with a choice $\tilde{e}_U \in \mathcal{M}_U$ lifting $\tilde{e}$ satisfying (5.65) is classified by a set of roots of unit

$$\{u_x \in \mathbf{k}^\times \mid u_x^{\mathbf{c}(x)} = 1, x \in \mathbf{E}(G)\}. \tag{5.71}$$

Thus we obtain the number of pairs

$$|\{(U, \tilde{e}_U) \mid U \in W \text{ and } \tilde{e}_U \text{ satisfies (5.65)}\}| = \prod_{x\in\mathbf{E}(G)} \mathbf{c}(x)$$

given by counting the number of sets (5.71). Further note that fixing a $U \in W$, there are precisely l different liftings $\tilde{e}_U$ satisfying (5.65). Thus the number of points of W is

$$\frac{\prod_{x\in \mathbf{E}(G)} \mathbf{c}(x)}{l} = \mu(\boldsymbol{\tau}_\curlywedge)$$

as in (5.12). This completes the proof of Proposition 5.8. $\square$

6. Reduction to connected invariants

In this section, we fix a $\curlywedge$-tropical type

$$\tau_\infty = (G_\infty, \mathbf{g}, \mathbf{deg}, \boldsymbol{\sigma}, \mathbf{c}, \mathbf{V}_{\max}(G_\infty) = \mathbf{V}(G_\infty)). \tag{6.1}$$

where $\mathbf{E}(G_\infty) = \emptyset$. We do not assume that τ_∞ is of compact type unless we work with the reduced theory. Furthermore, G_∞ is not necessarily connected. For each $V \in \mathbf{V}(G_\infty)$, denote by $G_V \subset G_\infty$ the connected component with $\mathbf{V}(G_V) = \{V\}$. Denote by τ_V the tropical type obtained by restricting τ_∞ to G_V. Note that we do not view τ_V as a $\curlywedge$-tropical type. For simplicity, we will write

$$\mathfrak{U}_V := \mathfrak{U}(\tau_V), \qquad \mathfrak{M}_V := \mathfrak{M}(\bar{\tau}_V).$$

For later use, denote by

$$e_{\max,V} \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{U}_V}, \mathfrak{U}_V), \qquad e_{\max,\infty} \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{U}_\infty}, \mathfrak{U}(\tau_\infty)).$$

the corresponding uniform maximal degeneracies of $\mathfrak{U}_V$ and $\mathfrak{U}(\tau_\infty)$. The same notations will be used to denote their pull-backs.

In this section we will develop an explicit formula expressing the fundamental class $[\mathfrak{U}(\tau_\infty)]$ using $[\mathfrak{U}_V]$, which will play an important role in both the tropical calculation of this paper and in the localization calculation [17].

6.1. Punctured maps with uniform minimal degeneracy.

6.1.1. *The log blow-up construction.*

Definition 6.1. A punctured map $\mathfrak{f}\colon C^\circ \to \infty_{\mathcal{A}}$ (with possibly disconnected domain) over an fs base S is said to have *uniform minimal degeneracy* if for each geometric point $s \in S$ the set of degeneracies of the geometric fiber $\mathfrak{f}_s$ has a unique minimum under the monoid partial ordering.

Consider a punctured map $\mathfrak{f}\colon C^\circ \to \infty_{\mathcal{A}}$ (with possibly disconnected domain) over an fs base S. For each geometric point $s \in S$, define the subset $\overline{\mathbf{D}}(\mathfrak{f}_s) \subset \overline{\mathcal{M}}_{S,s}$ of degeneracies of the geometric fiber $\mathfrak{f}_s$, and $\overline{\mathbf{D}}^m(\mathfrak{f}_s) \subset \overline{\mathbf{D}}(\mathfrak{f}_s)$ the subset of minimal degeneracies under the monoid partial ordering. Since $\infty_{\mathcal{A}}$ is the target of $\mathfrak{f}_s$, all components are degenerate. Hence we have $\overline{\mathbf{D}}(\mathfrak{f}_s) \neq \emptyset$ and $0 \notin \overline{\mathbf{D}}(\mathfrak{f}_s)$. This implies

$$\overline{\mathbf{D}}^m(\mathfrak{f}_s) \neq \emptyset, \quad \text{and} \quad 0 \notin \overline{\mathbf{D}}^m(\mathfrak{f}_s) \tag{6.2}$$

Since degeneracies and their partial orderings are stable under generization [20, §3.1.2], the sets $\overline{\mathbf{D}}(\mathfrak{f}_s)$ (resp. $\overline{\mathbf{D}}^m(\mathfrak{f}_s)$) for all geometric point $s \in S$ glue to a sub-sheaf of sets $\overline{\mathbf{D}}(\mathfrak{f}) \subset \overline{\mathcal{M}}_S$ (resp. $\overline{\mathbf{D}}(\mathfrak{f}_s) \subset \overline{\mathcal{M}}_S$), called the *sheaf of degeneracies* (resp. *sheaf of minimal degeneracies*). Thus if $\mathfrak{f}$ has the uniform minimal degeneracy, then the sheaf $\overline{\mathbf{D}}^m(\mathfrak{f})$ consists of a unique element $e_{\min} \in \Gamma(\overline{\mathcal{M}}_S, S)$ called the *minimal degeneracy* of $\mathfrak{f}$.

Denote by $\overline{\mathcal{K}}^m_S \subset \overline{\mathcal{M}}_S$ the sheaf of monoid ideals generated by $\overline{\mathbf{D}}^m(\mathfrak{f})$, called the *minimal log-ideal*. By (6.2), the monoid ideal $\overline{\mathcal{K}}^m_S$ is nowhere trivial. Denote by $\mathcal{K}^m_S \subset \mathcal{M}_S$ the log-ideal obtained by taking the pre-image of $\overline{\mathcal{K}}^m_S$ along the quotient $\mathcal{M}_S \to \overline{\mathcal{M}}_S$. Consider the log blow-up along the minimal log-ideal $\mathcal{K}^m_S$ in the fs category [31, III 2.6]:

$$\mathrm{Bl}\colon S^\curlyvee \to S. \tag{6.3}$$

Note that Bl is projective and log étale [31, III 2.6.4, 2.6.5]. Denote by $\mathfrak{f}^{\Upsilon}\colon (C^{\circ})^{\Upsilon} \to \infty_{\mathcal{A}}$ the punctured map over S^{Υ} obtained by pulling back $\mathfrak{f}$. It carries the following universal property.

Proposition 6.2. *Notations and assumptions as above, we have*

(1) $\mathfrak{f}^{\Upsilon}$ *is a punctured map over* S^{Υ} *with uniform minimal degeneracy.*

(2) Let $T \to S$ *be any morphism of fs log schemes, and* $\mathfrak{f}_T\colon C^{\circ}_T \to \infty_{\mathcal{A}}$ *be the punctured map over* T *obtained by pulling back* $\mathfrak{f}$*. If* $\mathfrak{f}_T$ *has uniform minimal degeneracy then* $T \to S$ *factorizes* Bl *uniquely such that* $\mathfrak{f}_T$ *is obtained by pulling back* $\mathfrak{f}^{\Upsilon}$*.*

In particular, if $\mathfrak{f}$ *has uniform maximal degeneracy then* Bl *is an identity.*

Proof. Consider a punctured map $\mathfrak{f}_T\colon C^{\circ}_T \to \infty_{\mathcal{A}}$ over T obtained by pulling back $\mathfrak{f}$ along $h\colon T \to S$. Observe that $\mathcal{K}^m_T = h^{\bullet}\mathcal{K}^m_S$. Thus $\mathfrak{f}_T$ has uniform minimal degeneracy implies that $\mathcal{K}^m_T$ is principal, hence h factors through Bl^m uniquely by the universal property of log blow-ups [31, III 2.6.1 (1)].

It suffices to verify that $\mathfrak{f}^{\Upsilon}$ has uniform minimal degeneracy. For each geometric point $s \in S^{\Upsilon}$, the monoid ideal $\overline{\mathcal{K}}^m_{S^{\Upsilon},s}$ is generated by an element $e \in \overline{\mathcal{M}}_{S^{\Upsilon}}$. By the local construction of log blow-ups [31, III 2.6.4], the observation $\overline{\mathcal{K}}^m_{S^{\Upsilon},s} = \mathrm{Bl}^{\bullet}\,\overline{\mathcal{K}}^m_{S,s}$ implies that this element e is a pull-back of an element in $\overline{\mathbf{D}}^m(\mathfrak{f})$ hence a degeneracy. Consequently, e is the minimal degeneracy over s as it generates other elements in $\overline{\mathcal{K}}^m_{S^{\Upsilon},s}$. This finishes the proof. ☐

6.1.2. *Vertex moduli with uniform minimal degeneracy.* For each $V \in \mathbf{V}(G_{\infty})$ consider the square

$$\begin{tikzcd} \mathfrak{U}^{\Upsilon}_V \arrow[r, "\mathrm{Bl}_V"] \arrow[d, "F^{\Upsilon}_V"'] & \mathfrak{U}_V \arrow[d, "F_V"] \\ \mathfrak{M}^{\Upsilon}_V \arrow[r, "\mathrm{Bl}_V"] & \mathfrak{M}_V \end{tikzcd} \tag{6.4}$$

where both horizontal arrows are defined as in (6.3) by taking log blow-ups of the corresponding minimal log-ideals, hence are both denoted by Bl_V by abuse of notations. By the functoriality of log blow-ups [31, III 2.6.3 (1)], Diagram (6.4) is Cartesian in the fs category. By Proposition 6.2, the stack $\mathfrak{M}^{\Upsilon}_V$ (resp. $\mathfrak{U}^{\Upsilon}_V$) parameterizes punctured maps marked by $\bar{\tau}_V$ with uniform minimal degeneracies (resp. marked by τ_V with both uniform minimal and uniform maximal degeneracies).

Observe that $\bar{\tau}_V$ is realizable. Thus there is an open dense substack $\mathfrak{M}^{\circ}_V \subset \mathfrak{M}_V$ parameterizing punctured maps with smooth irreducible domain curves. Thus the universal punctured map over $\mathfrak{M}^{\circ}_V$ has both uniform maximal and minimal degeneracies, which both equals the degeneracy of the unique component. In particular, all arrows in (6.4) restricting identities over $\mathfrak{M}^{\circ}_V$. As all arrows in (6.4) are proper and log étale, all stacks contain $\mathfrak{M}^{\circ}_V$ as open dense substacks. To summarize, we have the following observation.

Lemma 6.3. *All arrows in* (6.4) *are proper and birational. In particular, the push-forwards of the fundamental classes along the arrows in* (6.4) *are given by:*

$$\begin{tikzcd} {[\mathfrak{U}^{\Upsilon}_V]} \arrow[r, mapsto, "\mathrm{Bl}_{V,*}"] \arrow[d, mapsto, "F^{\Upsilon}_{V,*}"'] & {[\mathfrak{U}_V]} \arrow[d, mapsto, "F_{V,*}"] \\ {[\mathfrak{M}^{\Upsilon}_V]} \arrow[r, mapsto, "\mathrm{Bl}_{V,*}"] & {[\mathfrak{M}_V]} \end{tikzcd}$$

For later use, denote by

$$e_{\min,V} \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{M}^{\Upsilon}_V}, \mathfrak{M}^{\Upsilon}_V)$$

the minimal degeneracy of $\mathfrak{M}^{\Upsilon}_V$ and its pull-backs.

6.1.3. *G_∞-Moduli with uniform minimal degeneracy.* Consider the log blow-up

$$\mathrm{Bl}_\infty\colon \mathfrak{M}_\infty^{\Upsilon\Upsilon} := \Big(\prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}_V^{\Upsilon}\Big)^{\Upsilon} \to \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}_V^{\Upsilon} \tag{6.5}$$

along the minimal log-ideal $\mathcal{K}^m_{\prod_V \mathfrak{M}_V^{\Upsilon}}$. Denote by $\mathfrak{f}^{\Upsilon\Upsilon}\colon (C^\circ)^{\Upsilon\Upsilon} \to \infty_{\mathcal{A}}$ the universal punctured map over $\mathfrak{M}_\infty^{\Upsilon\Upsilon}$. There is a decomposition to connected components

$$(C^\circ)^{\Upsilon\Upsilon} = \bigsqcup_V (C^\circ)_V^{\Upsilon\Upsilon}, \qquad \mathfrak{f}_V^{\Upsilon\Upsilon} := \mathfrak{f}^{\Upsilon\Upsilon}|_{(C^\circ)_V^{\Upsilon\Upsilon}}\colon (C^\circ)_V^{\Upsilon\Upsilon} \to \infty_{\mathcal{A}}.$$

labeled by vertices $V \in \mathbf{V}(G_\infty)$ By Proposition 6.2, for each $V \in \mathbf{V}(G_\infty)$, $\mathfrak{f}_V^{\Upsilon\Upsilon}$ has the uniform minimal degeneracy $e_{\min,V}$. Furthermore, $\mathfrak{f}^{\Upsilon\Upsilon}$ has the uniform minimal degeneracy

$$e_{\min,\infty} \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{M}_\infty^{\Upsilon\Upsilon}}, \mathfrak{M}_\infty^{\Upsilon\Upsilon}).$$

As usual, we will also use $e_{\min,\infty}$ to denote its pull-backs.

Lemma 6.4. *The morphism Bl_∞ is given by a composition*

$$\mathfrak{M}_\infty^{\Upsilon\Upsilon} \xrightarrow{\mathrm{S}} (\textstyle\prod_V \mathfrak{M}_V^{\Upsilon})^{\Upsilon,f} \xrightarrow{\mathrm{Bl}_\infty^f} \textstyle\prod_V \mathfrak{M}_V^{\Upsilon} \tag{6.6}$$

such that

1. *Bl_∞^f is the log blow-up along $\mathcal{K}^m_{\prod_V \mathfrak{M}_V^{\Upsilon}}$ in the fine category. In particular, the underlying of Bl_∞^f is the projection*
$$\underline{(\prod_V \mathfrak{M}_V^{\Upsilon})^{\Upsilon,f}} \cong \mathbb{P}(\bigoplus_i \mathcal{O}(e_{\min,V})) \to \prod_V \underline{\mathfrak{M}_V^{\Upsilon}}, \tag{6.7}$$
where $\mathcal{O}(e_{\min,V})$ is the line bundle associated to $e_{\min,V}$, §A.2.2.
2. *S is the saturation which is an isomorphism over the open dense substack $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ} \subset \mathfrak{M}_\infty^{\Upsilon\Upsilon}$ with smooth domain curves.*
3. *There is an open dense substack $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ} \subset \mathfrak{M}_\infty^{\Upsilon\Upsilon\circ}$ such that the degeneracies of connected components are all identical.*

In particular, $\overline{\mathcal{M}}_{\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ}} \cong \mathbb{N}_{\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ}}$ is globally constant. Furthermore, the universal punctured map $\mathfrak{f}^{\Upsilon\Upsilon}|_{\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ}}$ admits both uniform minimal and maximal degeneracies, which coincide over $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ}$.

Proof. The factorization $\mathrm{Bl}_\infty = \mathrm{Bl}_\infty^f \circ \mathrm{S}$ follows from the construction of log blow-ups in the fs category [31, III 2.6.3]. Observe that the minimal monoid ideal $\mathcal{K}^m_{\prod_V \mathfrak{M}_V^{\Upsilon}}$ is generated by the set of minimal degeneracies $\{e_{\min,V} \mid V \in \mathbf{V}(G_\infty)\}$. Thus (3) follows from the local description [31, III 2.6.4] of log blow-ups.

Consider the open dense substack $\mathfrak{M}_V^\circ \subset \mathfrak{M}_V$ as in §6.1.2. Note that the characteristic sheaf $\overline{\mathcal{M}}_{\prod_V \mathfrak{M}_V^\circ}$ is locally free with generators given by $\{e_{\min,V}\}_V$. Let $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ}$ be the pre-image of $\prod_V \mathfrak{M}_V^\circ$ in $(\prod_V \mathfrak{M}_V^{\Upsilon})^{\Upsilon,f}$. A straightforward calculation shows that the saturation is trivial over $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ}$. Since log blow-ups are log étale, $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ}$ is open and dense in $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ}$, proving (2).

Finally, $\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ} \subset \mathfrak{M}_\infty^{\Upsilon\Upsilon\circ}$ is the open locus along which the sections $\{e_{\min,V}|_{\mathfrak{M}_\infty^{\Upsilon\Upsilon\circ\circ}}\}$ are identical, and are equal to $e_{\min,G_\infty}$. This implies (3). ☐

6.1.4. *G_∞ moduli with both uniform minimal and maximal degeneracies.* Consider the following Cartesian squares in the fs category

$$\begin{tikzcd}
\mathfrak{U}_\infty^{\Upsilon\Upsilon} \arrow[r, "\mathrm{Bl}_\infty"] \arrow[d, "F_\infty^{\Upsilon\Upsilon}"'] & \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{U}_V^{\Upsilon} \arrow[r, "\prod \mathrm{Bl}_V"] \arrow[d, "\prod F_V^{\Upsilon}"] & \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{U}_V \arrow[d, "\prod F_V"] \\
\mathfrak{M}_\infty^{\Upsilon\Upsilon} \arrow[r, "\mathrm{Bl}_\infty"] & \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}_V^{\Upsilon} \arrow[r, "\prod \mathrm{Bl}_V"] & \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}_V
\end{tikzcd} \tag{6.8}$$

where the square on the right is given by the product of (6.4). By the base change of log blow-ups [31, III 2.6.3 (1)], the top horizontal arrows are again log blow-ups along the corresponding minimal log-ideals, hence are labeled by the same notations $\mathrm{Bl}_\infty, \prod \mathrm{Bl}_V$ as in the bottom. Since the horizontal arrows of (6.8) are log étale, Lemma 6.3 and Lemma 6.4 implies that $\mathfrak{U}_\infty^{\curlyvee\curlyvee}$ is equidimensional of

$$\dim \mathfrak{U}_\infty^{\curlyvee\curlyvee} = \sum_{V \in \mathbf{V}(G_\infty)} \dim \mathfrak{M}_V + |\mathbf{V}(G_\infty)| - 1. \tag{6.9}$$

Denote by $\mathfrak{f}^\diamond \colon (C^\circ)^\diamond \to \infty_{\mathcal{A}}$ the universal punctured map over $\mathfrak{U}_\infty^{\curlyvee\curlyvee}$. There is a decomposition to connected components

$$(C^\circ)^\diamond = \bigsqcup_V (C^\circ)^\diamond_V, \qquad \mathfrak{f}^\diamond_V := \mathfrak{f}^\diamond|_{(C^\circ)^\diamond_V} \colon (C^\circ)^\diamond_V \to \infty_{\mathcal{A}}. \tag{6.10}$$

labeled by vertices $V \in \mathbf{V}(G_\infty)$. For each V, as $\mathfrak{f}^\diamond_V$ is the pull-back of the universal punctured map over $\mathfrak{U}_V^{\curlyvee}$, it admits both uniform minimal and maximal degeneracies $e_{\min,V}$ and $e_{\max,V}$ respectively. Furthermore, $\mathfrak{f}^\diamond$ as the pull-back of $\mathfrak{f}^{\curlyvee\curlyvee}$ admits a uniform minimal degeneracy $e_{\min,\infty}$.

6.2. **Punctured maps with partially aligned maximal degeneracies.** For a non-empty subset $\mathbf{V} \subset \mathbf{V}(G_\infty)$, consider the subgraph $G_{\mathbf{V}} = \sqcup_{V \in \mathbf{V}} G_V$. Recall that $G_V \subset G_\infty$ is the connected component with $\mathbf{V}(G_V) = \{V\}$. Let $\tau_{\mathbf{V}}$ be the $\curlywedge$-tropical type by restricting τ_∞ to $G_{\mathbf{V}}$ with $\mathbf{V}_{\max}(G_{\mathbf{V}}) = \mathbf{V}(G_{\mathbf{V}})$. Let $\mathbf{V}^c = \mathbf{V}(G_\infty) \setminus \mathbf{V}$ be the complement. Write for simplicity

$$\mathfrak{U}_{\mathbf{V}} := \mathfrak{U}(\tau_{\mathbf{V}}), \qquad \mathfrak{U}_{\mathbf{V}|\mathbf{V}^c} := \mathfrak{U}_{\mathbf{V}} \times \prod_{V \in \mathbf{V}^c} \mathfrak{U}_V.$$

Then over $\mathfrak{U}_{\mathbf{V}|\mathbf{V}^c}$ we have a collection of maximal degeneracies

$$e_{\max,\mathbf{V}}, \qquad \{e_{\max,V}\}_{V \in \mathbf{V}^c} \tag{6.11}$$

by pulling back the maximal degeneracies of the corresponding components $\mathfrak{U}_{\mathbf{V}}$ and $\mathfrak{U}_V$.

Observe that $\mathfrak{U}_{\mathbf{V}|\mathbf{V}^c} = \mathfrak{U}(\tau_\infty)$ if $\mathbf{V} = \mathbf{V}(G_\infty)$, and $\mathfrak{U}_{\mathbf{V}|\mathbf{V}^c} = \prod_{V \in \mathbf{V}(G_\infty)} \mathfrak{U}_V$ if $|\mathbf{V}| = 1$. However, there are no morphisms between $\mathfrak{U}_{\mathbf{V}|\mathbf{V}^c}$ for different choices of $\mathbf{V}$ unless $|\mathbf{V}| = 1$. We fix this issue by consider alignments of maximal degeneracies of vertices in $\mathbf{V}$ as follows.

Consider the set of maximal degeneracies

$$\overline{\mathbf{D}}^M_{\mathbf{V}} = \{e_{\max,V} \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{U}_\infty^{\curlyvee\curlyvee}}, \mathfrak{U}_\infty^{\curlyvee\curlyvee})\}_{V \in \mathbf{V}}$$

over $\mathfrak{U}_\infty^{\curlyvee\curlyvee}$. Denote by $\mathfrak{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a|\mathbf{V}^c} := \mathcal{T}_{\overline{\mathbf{D}}^M_{\mathbf{V}}}$ the stack of log alignments of $\overline{\mathbf{D}}^M_{\mathbf{V}}$ as in §C.2.2. Pulling back universal objects along the tautological morphism

$$\mathfrak{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a|\mathbf{V}^c} \to \mathfrak{U}_\infty^{\curlyvee\curlyvee}, \tag{6.12}$$

we obtain the universal punctured map $\mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c} \colon (C^\circ)_{\mathbf{V}|\mathbf{V}^c} \to \infty_{\mathcal{A}}$ over $\mathfrak{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a|\mathbf{V}^c}$. Further pulling back (6.10), there is a decomposition to connected components

$$(C^\circ)_{\mathbf{V}^a|\mathbf{V}^c} = \bigsqcup_{V \in \mathbf{V}(G_\infty)} (C^\circ)_{\mathbf{V}^a|\mathbf{V}^c,V}, \ \mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c,V} := \mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c}|_{(C^\circ)_{\mathbf{V}^a|\mathbf{V}^c,V}} \colon (C^\circ)_{\mathbf{V}^a|\mathbf{V}^c,V} \to \infty_{\mathcal{A}}.$$

Moreover by §C.2.1, the set of maximal degeneracies $(\overline{\mathbf{D}}^M_{\mathbf{V}})|_s = \{e_{\max,V}|_s\}_{V \in \mathbf{V}}$ over each geometric point $s \in \mathfrak{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a|\mathbf{V}^c}$ is totally ordered. Thus the punctured map with disconnected domain

$$\mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c,\mathbf{V}} := \bigsqcup_{V \in \mathbf{V}} \mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c,V} \colon (C^\circ)_{\mathbf{V}^a|\mathbf{V}^c,\mathbf{V}} = \bigsqcup_{V \in \mathbf{V}} (C^\circ)_{\mathbf{V}^a|\mathbf{V}^c,V} \to \infty_{\mathcal{A}}$$

has the uniform maximal degeneracy. This induces a natural morphism

$$\mathfrak{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a|\mathbf{V}^c} \to \mathfrak{U}_{\mathbf{V}|\mathbf{V}^c}, \tag{6.13}$$

where the maximal degeneracies $e_{\max,V}, e_{\max,\mathbf{V}}$ as in (6.11) pulls back to maximal degeneracies of $\mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c,V}$ and $\mathfrak{f}_{\mathbf{V}^a|\mathbf{V}^c,\mathbf{V}}$ over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}$ respectively.

Now consider a sequence of non-empty subsets $\mathbf{V}(G_\infty) \supset \mathbf{V}_1 \supset \mathbf{V}_2$. By (C.7), we obtain a tautological morphism over $\mathfrak{U}^{\Upsilon\Upsilon}_\infty$

$$\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2} \colon \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c} \to \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}. \tag{6.14}$$

Lemma 6.5. *The morphism $\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2}$ is log étale, projective and birational. In particular for any non-empty subset $\mathbf{V} \subset \mathbf{V}(G_\infty)$, the stack $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}$ is equidimensional of*

$$\dim \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c} = \sum_{V\in\mathbf{V}(G_\infty)} \dim \mathfrak{M}_V + |\mathbf{V}(G_\infty)| - 1. \tag{6.15}$$

Proof. The log étaleness and projectivity of $\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2}$ follow from Proposition C.5. By Lemma 6.4 (3), both the domain and target of $\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2}$ contain $\mathfrak{M}^{\Upsilon\Upsilon\circ\circ}_\infty$ as an open and dense substack. This proves the birationality of $\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2}$. The dimension calculation follows from (6.9). ☐

Consider the case that $\mathbf{V} = \mathbf{V}(G_\infty)$, hence $\mathbf{V}^c = \emptyset$. Then over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset}$, the set of maximal degeneracies $\{e_{\max,V}\}_{V\in\mathbf{V}(G_\infty)}$ are aligned. In particular, there is a tautological morphism

$$\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset} \to \mathfrak{U}(\tau_\infty) \tag{6.16}$$

Lemma 6.6. *The morphism (6.16) is proper and birational.*

Proof. Indeed, we have a tautological commutative diagram

$$\begin{tikzcd}
\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset} \arrow[rr] \arrow[dr] & & \mathfrak{U}(\tau_\infty) \arrow[dl] \\
& \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}_V &
\end{tikzcd}$$

The left downward arrow is projective by the projectivity of log blows, (6.8) and Lemma 6.5. The right downward arrow is proper and representable by Proposition 3.13. This implies the properness of the horizontal arrow.

To see the birationality, observe that the same stack $\mathfrak{M}^{\Upsilon\Upsilon\circ\circ}_\infty$ in Lemma 6.4 (3) is open and dense in both the domain and target of (6.16). ☐

6.3. **Comparing maximal degeneracies.** Consider non-empty subsets $\mathbf{V}(G_\infty) \supset \mathbf{V}_1 \supset \mathbf{V}_2$ satisfying $\mathbf{V}_1 \setminus \mathbf{V}_2 = \{V\}$. We want to compare the three maximal degeneracies $e_{\max,\mathbf{V}_1}, e_{\max,\mathbf{V}_2}$ and $e_{\max,V}$ over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$. where $e_{\max,\mathbf{V}_2}$ and $e_{\max,V}$ are obtained by pulling back the corresponding degeneracies over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}$ via (6.14). Observe that

$$e_{\max,\mathbf{V}_1,s} = \max(e_{\max,\mathbf{V}_2,s}, e_{\max,V,s}) \tag{6.17}$$

at every geometric point $s \in \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$. We thus obtain a global section

$$\delta_{\mathbf{V}_1\supset\mathbf{V}_2} := (e_{\max,V} + e_{\max\mathbf{V}_2} - e_{\max,\mathbf{V}_1}) \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}, \mathfrak{U}_{\mathbf{V}_1^a|\mathbf{V}_1^c}), \tag{6.18}$$

measuring the differences between the degeneracies. As $\delta_{\mathbf{V}_1\supset\mathbf{V}_2}$ plays a key role in the reduction of reduced virtual cycles §6.6, we now study its structure as follows.

Consider following two global sections

$$(e_{\max,\mathbf{V}_1} - e_{\max,\mathbf{V}_2}), (e_{\max,\mathbf{V}_1} - e_{\max,V}) \in \Gamma(\overline{\mathcal{M}}_{\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}, \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}).$$

By (A.2), we obtain canonical morphisms of line bundles over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$

$$\mathcal{O}(-e_{\max,\mathbf{V}_1}) \longrightarrow \mathcal{O}(-e_{\max,\mathbf{V}_2}), \qquad \mathcal{O}(-e_{\max,\mathbf{V}_1}) \longrightarrow \mathcal{O}(-e_{\max,V}) \tag{6.19}$$

This induces a morphism of line bundles over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1|\mathbf{V}_1^c}$

$$\mathcal{O}(-e_{\max,\mathbf{V}_1}) \longrightarrow \mathcal{O}(-e_{\max,V}) \oplus \mathcal{O}(-e_{\max,\mathbf{V}_2}). \tag{6.20}$$

Lemma 6.7. *The morphism* (6.20) *is injective, with cokernel* $\mathcal{O}(-\delta_{\mathbf{V}_1\supset\mathbf{V}_2})$.

Proof. It suffices to show that fiberwise over $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$ the two morphisms in (6.19) cannot vanish simultaneously. By (6.17) over each geometric point $s \to \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$ one has at least one of the following

$$(e_{\max,\mathbf{V}_1} - e_{\max,\mathbf{V}_2})|_s = 0 \quad \text{or} \quad (e_{\max,\mathbf{V}_1} - e_{\max,V})|_s = 0.$$

This implies that at least one of the two morphisms in (6.19) is injective over s. This proves the statement. □

Recall the minimal degeneracies

$$e_{\min,\infty} \in \Gamma\left(\overline{\mathcal{M}}_{\mathfrak{U}^{\Upsilon\Upsilon}_{\infty}}, \mathfrak{U}^{\Upsilon\Upsilon}_{\infty}\right) \quad \text{and} \quad e_{\min,V} \in \Gamma\left(\overline{\mathcal{M}}_{\mathfrak{M}^{\Upsilon}_{V}}, \mathfrak{M}^{\Upsilon}_{V}\right).$$

We introduce the tautological Chern classes

$$\psi_{\min} := c_1\left(\mathcal{O}(-e_{\min,\infty})\right) \quad \text{and} \quad \psi_{\min,V} := c_1\left(\mathcal{O}(-e_{\min,V})\right). \tag{6.21}$$

The same notations will be used to denote their pull-backs.

Lemma 6.8. $\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*} c_1\left(\mathcal{O}(\delta_{\mathbf{V}_1\supset\mathbf{V}_2})\right) = -\psi_{\min}$.

Proof. Consider the divisor $\mathbf{D} := c_1\left(\mathcal{O}(\delta_{\mathbf{V}_1\supset\mathbf{V}_2})\right) + \psi_{\min}$ on $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$. It suffices to show that $\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}\,\mathbf{D} = 0$. By (6.18) we have a relation

$$c_1\left(\mathcal{O}(\delta_{\mathbf{V}_1\supset\mathbf{V}_2})\right) = c_1(\mathcal{O}(e_{\max,V})) + c_1(\mathcal{O}(e_{\max,\mathbf{V}_2})) - c_1(\mathcal{O}(e_{\max,\mathbf{V}_1}))$$

on $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$. Write

$$\mathbf{D} = \mathbf{D}_1 - \mathbf{D}_2 := c_1\big(\mathcal{O}(e_{\max,V} - e_{\min,\infty})\big) - c_1\big(\mathcal{O}(e_{\max,\mathbf{V}_1} - e_{\max,\mathbf{V}_2})\big).$$

We observe that $\mathbf{D}_1$ and $\mathbf{D}_2$ can be represented by effective Cartier divisors. Indeed, the relations

$$e_{\min,\infty} \preccurlyeq e_{\max,\mathbf{V}_2} \preccurlyeq e_{\max,\mathbf{V}_1}, \qquad e_{\min,\infty} \preccurlyeq e_{\max,V} \preccurlyeq e_{\max,\mathbf{V}_1},$$

implies that $(e_{\max,\mathbf{V}_1} - e_{\max,\mathbf{V}_2})$ and $(e_{\max,V} - e_{\min,\infty})$ are global sections of $\overline{\mathcal{M}}_{\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}$. Furthermore, they vanish over the open dense substack of $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$ defined by

$$e_{\max,\mathbf{V}_1} = e_{\max,V} = e_{\max,\mathbf{V}_2} = e_{\min,\infty}.$$

Hence, $\mathbf{D}_1$ and $\mathbf{D}_2$ are identified with effective Cartier divisors as in (A.4), supported along the loci satisfying the following conditions respectively

$$0 \prec (e_{\max,V} - e_{\min,\infty}), \qquad 0 \prec (e_{\max,\mathbf{V}_1} - e_{\max,\mathbf{V}_2}).$$

By (6.17), $\mathbf{D}_2$ is supported along the locus $e_{\max,\mathbf{V}_2} \prec e_{\max,V}$. Also note that $\mathbf{D}_1$ is supported along the locus $e_{\min,\infty} \prec e_{\max,V}$. Since $e_{\min,\infty} \prec e_{\max,V}$, we conclude that $\mathbf{D} = \mathbf{D}_1 - \mathbf{D}_2$ is represented by an effective divisor in $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$.

Denote by $\mathbf{D}_1' \subset \mathbf{D}_1$ and $\mathbf{D}_2' \subset \mathbf{D}_2$ the components whose generic points are supported along the locus $e_{\max,\mathbf{V}_2} = e_{\min,\infty}$. The local description in §A.2.2 implies that $\mathbf{D}_1' = \mathbf{D}_2'$. Thus, this effective divisor $\mathbf{D} = \mathbf{D}_1 - \mathbf{D}_2$ is supported along the locus given by

$$e_{\min,\infty} \prec e_{\max,\mathbf{V}_2} \prec e_{\max,V}. \tag{6.22}$$

Finally the two strict inequality in (6.22) implies that the image of this locus in $\mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}$ is of codimension at least 2, hence $\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}\,\mathbf{D} = 0$ as desired. □

6.4. **Splitting of $\psi_{\min}$.** Lemma 6.8 shows that on the cycle level the differences of maximal degeneracies (6.18) is given by the Chern class $\psi_{\min}$ defined on $\mathfrak{U}^{\Upsilon\Upsilon}_{\infty}$. To further push-forward $\psi_{\min}$ along $\mathrm{Bl}_{\infty}\colon \mathfrak{U}^{\Upsilon\Upsilon}_{\infty} \to \prod_{V\in\mathbf{V}(G_{\infty})}\mathfrak{U}^{\Upsilon}_{V}$ in (6.8), set

$$\frac{1}{-t-\psi_{\min}} = -\frac{1}{t}\Big(1+\frac{\psi_{\min}}{-t}+(\frac{\psi_{\min}}{-t})^2+\cdots\Big). \tag{6.23}$$

where t is a formal parameter. We compute that

Lemma 6.9. $\mathrm{Bl}_{\infty,*}\left(\frac{1}{-t-\psi_{\min}}\right) = \prod_{V\in\mathbf{V}(G_{\infty})}\frac{1}{-t-\psi_{\min,V}}$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{U}^{\Upsilon\Upsilon}_{\infty} \arrow[r, "\mathrm{S}"] \arrow[d, "F^{\diamond}_{\infty}"'] & (\prod_V \mathfrak{U}^{\Upsilon}_{V})^{\Upsilon,f} \arrow[r, "\mathrm{Bl}^{f}_{\infty}"] \arrow[d] & \prod_{V\in\mathbf{V}(G_{\infty})}\mathfrak{U}^{\Upsilon}_{V} \arrow[d, "\prod F_V"] \\
\mathfrak{M}^{\Upsilon\Upsilon}_{\infty} \arrow[r, "\mathrm{S}"] & (\prod_V \mathfrak{M}^{\Upsilon}_{V})^{\Upsilon,f} \arrow[r, "\mathrm{Bl}^{f}_{\infty}"] & \prod_{V\in\mathbf{V}(G_{\infty})}\mathfrak{M}^{\Upsilon}_{V}
\end{tikzcd} \tag{6.24}$$

where the bottom arrows are from (6.6), the right side square is Cartesian in the fine category, and the arrows labeled by S are the saturation. Since Bl^{f}_{∞} on the bottom is the log blow-up with respect to $\mathcal{K}^{m}_{\prod_V \mathfrak{M}^{\Upsilon}_{V}}$. The functoriality of log blow-ups implies that Bl^{f}_{∞} on the top is the log blow-up with respect to $\mathcal{K}^{m}_{\prod_V \mathfrak{U}^{\Upsilon}_{V}} = (\prod F_V)^{\bullet}\mathcal{K}^{m}_{\prod_V \mathfrak{M}^{\Upsilon}_{V}}$. Thus pulling back (6.7), we obtain

$$\mathrm{Bl}^{f}_{\infty}\colon \underline{(\prod \mathfrak{U}^{\Upsilon}_{V})^{\Upsilon,f}} \cong \mathbb{P}(\bigoplus_{i}\mathcal{O}(e_{\min,V})) \to \prod_{V}\underline{\mathfrak{U}^{\Upsilon}_{V}}, \tag{6.25}$$

Note that the minimal degeneracy $e_{\min,\infty}$ exists as a global section of $\overline{\mathcal{M}}_{(\prod_V \mathfrak{U}^{\Upsilon}_{V})^{\Upsilon,f}}$, which pulls back to $e_{\min,\infty}$ over $\mathfrak{U}^{\Upsilon\Upsilon}_{\infty}$. Hence the line bundle $\mathcal{O}(-e_{\min,\infty})$ and the class $\psi_{\min}$ exist over $(\prod_V \mathfrak{U}^{\Upsilon}_{V})^{\Upsilon,f}$, which pull back to $\mathcal{O}(-e_{\min,\infty})$ and $\psi_{\min}$ over $\mathfrak{U}^{\Upsilon\Upsilon}_{\infty}$. Thus, we have $\mathrm{Bl}_{\infty,*}\frac{1}{-t-\psi_{\min}} = \mathrm{Bl}^{f}_{\infty,*}\frac{1}{-t-\psi_{\min}}$.

To prove the statement, set $s_{1/t}(\bigoplus_V \mathcal{O}(e_{\min,V}))$ and $c_{1/t}(\bigoplus_V \mathcal{O}(e_{\min,V}))$ be the Segre and Chern polynomial of $\bigoplus_V \mathcal{O}(e_{\min,V})$ with variable $1/t$. Observe that $\mathcal{O}(-e_{\min,\infty})$ is the twisting sheaf $\mathcal{O}(1)$ of the projective bundle (6.25). Pushing forward (6.23), we have

$$\begin{aligned}
\mathrm{Bl}^{f}_{\infty,*}\frac{1}{-t-\psi_{\min}} &= -\frac{1}{t}\,\mathrm{Bl}^{f}_{\infty,*}\,\big((\frac{\psi_{\min}}{-t})^{m-1}+(\frac{\psi_{\min}}{-t})^{m}+\cdots\big) \\
&= \frac{(-1)^m}{t^m}\,\mathrm{Bl}^{f}_{\infty,*}\,\big(\psi^{m-1}_{\min}+\frac{\psi^{m}_{\min}}{-t}+\frac{\psi^{m+1}_{\min}}{(-t)^2}+\cdots\big) \\
&= \frac{(-1)^m}{t^m}s_{-1/t}\Big(\bigoplus_{V}\mathcal{O}(e_{\min,V})\Big) \\
&= \frac{(-1)^m}{t^m}c^{-1}_{-1/t}\Big(\bigoplus_{V}\mathcal{O}(e_{\min,V})\Big) \\
&= \frac{(-1)^m}{t^m}\prod_{V}\frac{1}{1+\psi_{\min,V}/t} \\
&= \prod_{V}\frac{1}{-t-\psi_{\min,V}}.
\end{aligned}$$

This proves the statement. ☐

6.5. **Reduction of the canonical virtual cycles.** Consider a decorated $\curlywedge$-tropical type

$$\boldsymbol{\tau}_\infty = (\tau_\infty, \overline{\gamma}_\infty, \boldsymbol{\beta}_\infty), \tag{6.26}$$

where τ_∞ is the $\curlywedge$-tropical type as in 6.1. For the purposes of the canonical theory, we do not assume the compactness of $\boldsymbol{\tau}_\infty$ in this section.

Let $\boldsymbol{\tau}_V$ be the decorated tropical type by restricting $\boldsymbol{\tau}_\infty$ to the subgraph G_V. For simplicity, we write

$$\mathscr{U}(\boldsymbol{\tau}_\infty) := \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_\infty), \qquad \mathscr{U}_V := \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_V), \qquad \mathscr{R}_V := \mathscr{R}(\mathfrak{P}, \boldsymbol{\tau}_V).$$

for the stacks of stable punctured R-maps.

By (6.4), we obtain Cartesian squares in the fs category with strict vertical arrows

$$\begin{tikzcd}
\mathscr{U}_V^{\curlyvee} \arrow[r, "\mathrm{Bl}_V"] \arrow[d, "\mathbf{ev}_V"] & \mathscr{U}_V \arrow[r, "F_{\mathscr{R}}"] \arrow[d] & \mathscr{R}_V \arrow[d] \\
\mathfrak{U}_V^{\curlyvee,\mathrm{ev}} \arrow[r] \arrow[d] & \mathfrak{U}_V^{\mathrm{ev}} \arrow[r] \arrow[d] & \mathfrak{M}_V^{\mathrm{ev}} \arrow[d] \\
\mathfrak{U}_V^{\curlyvee} \arrow[r, "\mathrm{Bl}_V"] & \mathfrak{U}_V \arrow[r, "F_{\mathfrak{M}}"] & \mathfrak{M}_V
\end{tikzcd} \tag{6.27}$$

where by abuse of notations we denote both top and bottom horizontal arrows by the same notation Bl_V. The stack $\mathscr{U}_V^{\curlyvee}$ parameterizes stable punctured R-maps with both uniform maximal and uniform minimal degeneracies with discrete data specified by $\boldsymbol{\tau}_V$. Pulling back the canonical perfect obstruction of $\mathscr{R}_V \to \mathfrak{M}_V^{\mathrm{ev}}$ as in (2.56), we obtain the canonical perfect obstruction theories for $\mathscr{U}_V^{\curlyvee} \to \mathfrak{U}_V^{\curlyvee,\mathrm{ev}}$ and $\mathscr{U}_V \to \mathfrak{U}_V^{\mathrm{ev}}$. By Lemma 6.3, these perfect obstruction theories define canonical virtual cycles $[\mathscr{U}_V^{\curlywedge}]^{\mathrm{vir}}, [\mathscr{U}_V]^{\mathrm{vir}}$ and $[\mathscr{R}_V]^{\mathrm{vir}}$, which satisfy the virtual push-forwards

$$[\mathscr{U}_V^{\curlyvee}]^{\mathrm{vir}} \xmapsto{\mathrm{Bl}_{V,*}} [\mathscr{U}_V]^{\mathrm{vir}} \xmapsto{F_{V,*}} [\mathscr{R}_V]^{\mathrm{vir}}. \tag{6.28}$$

For a non-empty subset $\mathbf{V} \subset \mathbf{V}(G_\infty)$, denote by $\boldsymbol{\tau}_{\mathbf{V}}$ the decorated $\curlywedge$-tropical type by restricting $\boldsymbol{\tau}_\infty$ to $G_{\mathbf{V}}$ with $\mathbf{V}_{\max}(G_{\mathbf{V}}) = \mathbf{V}$. In particular, the corresponding $\curlywedge$-tropical type of $\boldsymbol{\tau}_{\mathbf{V}}$ is $\tau_{\mathbf{V}}$ in §6.2. For simplicity, we write

$$\mathscr{U}_{\mathbf{V}} := \mathscr{U}(\mathfrak{P}, \boldsymbol{\tau}_{\mathbf{V}}), \qquad \mathscr{U}_{\mathbf{V}|\mathbf{V}^c} := \mathscr{U}_{\mathbf{V}} \times \prod_{V \in \mathbf{V}^c} \mathscr{U}_V. \tag{6.29}$$

Note that $\mathscr{U}_{\mathbf{V}} = \mathscr{U}_V$ and $\mathfrak{U}_{\mathbf{V}}^{\mathrm{ev}} = \mathfrak{U}_V^{\mathrm{ev}}$ when $\mathbf{V} = \{V\}$.

Consider the diagram of Cartesian squares with strict vertical arrows

$$\begin{tikzcd}
\mathscr{U}_{\mathbf{V}^a|\mathbf{V}^c}^{\curlyvee\curlyvee} \arrow[r] \arrow[d] & \mathscr{U}_{\mathbf{V}|\mathbf{V}^c} \arrow[r] \arrow[d] & \prod_{V \in \mathbf{V}(G_\infty)} \mathscr{R}_V \arrow[d] \\
\mathfrak{U}_{\mathbf{V}^a|\mathbf{V}^c}^{\mathrm{ev},\curlyvee\curlyvee} \arrow[r] \arrow[d] & \mathfrak{U}_{\mathbf{V}|\mathbf{V}^c}^{\mathrm{ev}} \arrow[r] \arrow[d] & \prod_{V \in \mathbf{V}(G_\infty)} \mathfrak{M}_V^{\mathrm{ev}} \arrow[d] \\
\mathfrak{U}_{\mathbf{V}^a|\mathbf{V}^c}^{\curlyvee\curlyvee} \arrow[r, "(6.13)"] & \mathfrak{U}_{\mathbf{V}|\mathbf{V}^c} \arrow[r] & \prod_{V \in \mathbf{V}(G_\infty)} \mathfrak{M}_V
\end{tikzcd} \tag{6.30}$$

The canonical perfect obstruction theory of $\prod_{V \in \mathbf{V}(G_\infty)} \mathscr{R}_V \to \prod_{V \in \mathbf{V}(G_\infty)} \mathfrak{M}_V^{\mathrm{ev}}$ as in (2.56) pulls back to the canonical perfect obstruction theory of $\mathscr{U}_{\mathbf{V}^a|\mathbf{V}^c}^{\curlyvee\curlyvee} \to \mathfrak{U}_{\mathbf{V}^a|\mathbf{V}^c}^{\mathrm{ev},\curlyvee\curlyvee}$. This defines the canonical virtual cycle $[\mathscr{U}_{\mathbf{V}^a|\mathbf{V}^c}^{\curlyvee\curlyvee}]^{\mathrm{vir}}$ by the equidimensionality in Lemma 6.5.

Now consider a sequence of non-empty subsets $\mathbf{V}(G_\infty) \supset \mathbf{V}_1 \supset \mathbf{V}_2$. We obtain another diagram of Cartesian squares with strict horizontal arrows

$$\begin{tikzcd}
\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c} \arrow[r] \arrow[d, "\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2}"'] & \mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c} \arrow[r] \arrow[d] & \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c} \arrow[d, "\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2}"] \\
\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c} \arrow[r] & \mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c} \arrow[r] & \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}
\end{tikzcd} \tag{6.31}$$

As both arrows admit the canonical perfect obstruction theories pulled back from $\prod_{V\in\mathbf{V}(G_\infty)} \mathscr{R}_V \to \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}^{\mathrm{ev}}_V$, Lemma 6.5 implies the virtual push-forward

$$\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{vir}} = [\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}]^{\mathrm{vir}}. \tag{6.32}$$

Now we consider the following diagram of Cartesian squares

$$\begin{tikzcd}
 & \mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset} \arrow[r, "\mathrm{align}"] \arrow[d] \arrow[dl, "F_{\mathscr{U}}"'] & \mathscr{U}^{\Upsilon\Upsilon}_{\infty} \arrow[r, "\mathrm{Bl}_\infty"] \arrow[d] & \prod_{V\in\mathbf{V}(G_\infty)} \mathscr{U}^{\Upsilon}_V \arrow[d] \\
\mathscr{U}(\boldsymbol{\tau}_\infty) \arrow[d] & \mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset} \arrow[r] \arrow[d] \arrow[dl] & \mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\infty} \arrow[r] \arrow[d] & \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{U}^{\mathrm{ev},\Upsilon}_V \arrow[d] \\
\mathfrak{U}^{\mathrm{ev}}(\tau_\infty) \arrow[d] & \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset} \arrow[r, "(6.14)"'] \arrow[dl, "(6.16)"] & \mathfrak{U}^{\Upsilon\Upsilon}_{\infty} \arrow[r] & \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{U}^{\Upsilon}_V \\
\mathfrak{U}(\tau_\infty) & & &
\end{tikzcd} \tag{6.33}$$

where the bottom right horizontal arrow is from (6.8), and $\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_\infty \cong \mathfrak{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}$ for any $\mathbf{V}$ with $|\mathbf{V}| = 1$.

While there is no morphism between $\mathscr{U}(\boldsymbol{\tau}_\infty)$ and $\prod_{V\in\mathbf{V}(G_\infty)} \mathscr{U}^{\Upsilon}_V$, their canonical virtual cycles are related via $[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{vir}}$ as follows.

Theorem 6.10. *We have the following virtual push-forwards along arrows in* (6.33)*:*

$$F_{\mathscr{R},*}\,\mathrm{Bl}_{V,*}[\mathscr{U}^{\Upsilon}_V]^{\mathrm{vir}} = [\mathscr{R}_V]^{\mathrm{vir}}, \qquad \forall \quad V \in \mathbf{V}(G_\infty). \tag{6.34}$$

$$F_{\mathscr{U},*}[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{vir}} = [\mathscr{U}(\boldsymbol{\tau}_\infty)]^{\mathrm{vir}}, \tag{6.35}$$

$$\mathrm{Bl}_{\infty,*}\,\mathrm{align}_*\left(\frac{[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{vir}}}{-t-\psi_{\min}}\right) = \prod_{V\in\mathbf{V}(G_\infty)} \frac{[\mathscr{U}^{\Upsilon}_V]^{\mathrm{vir}}}{-t-\psi_{\min,V}}. \tag{6.36}$$

Proof. The push-forward (6.34) is (6.28).

Note that the perfect obstruction theories of the four top vertical arrows in (6.33) are all pulled back from $\prod_{V\in\mathbf{V}(G_\infty)} \mathscr{R}_V \to \prod_{V\in\mathbf{V}(G_\infty)} \mathfrak{M}^{\mathrm{ev}}_V$. Thus the virtual push-forward (6.35) follows from Lemma 6.6.

Similarly applying Lemma 6.5, we obtain $\mathrm{align}_*[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{vir}} = [\mathscr{U}^{\Upsilon\Upsilon}_\infty]^{\mathrm{vir}}$. Further applying the projection formula, we obtain the virtual push-forward

$$\mathrm{align}_*\left(\frac{[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{vir}}}{-t-\psi_{\min}}\right) = \frac{[\mathscr{U}^{\Upsilon\Upsilon}_\infty]^{\mathrm{vir}}}{-t-\psi_{\min}}.$$

It remains to show that $\mathrm{Bl}_{\infty,*}\left(\frac{[\mathscr{U}_{\infty}^{\Upsilon\Upsilon}]^{\mathrm{vir}}}{-t-\psi_{\min}}\right) = \prod_V \frac{[\mathscr{U}_V^{\Upsilon}]^{\mathrm{vir}}}{-t-\psi_{\min,V}}$. This follows from Lemma 6.9 and the projection formula of virtual pull-backs [30, Theorem 4.1]. ☐

Remark 6.11. Taking the t^0-term from both sides of (6.36), we observe that

$$\mathrm{Bl}_{\infty,*}\,\mathrm{align}_*[\mathscr{U}_0^{\Upsilon\Upsilon}]^{\mathrm{vir}} = 0.$$

Thus (6.28) and (6.36) imply that further pushing forward along $F\colon \mathscr{U}(\boldsymbol{\tau}_\infty) \to \prod_V \mathscr{R}_V$, we obtain $F_*[\mathscr{U}(\boldsymbol{\tau}_\infty)]^{\mathrm{vir}} = 0$. However, $t^{<0}$-terms do not vanish in general, and lead to non-trivial localization contributions in the canonical theory. This will be investigated in [17].

6.6. **Reduction of the reduced virtual cycles.**

6.6.1. *Statement of the reduction.* For the reduced theory, we now assume that $\boldsymbol{\tau}_\infty$ hence τ_∞ are of compact type. This implies that for a non-empty subset $\mathbf{V} \subset \mathbf{V}(G_\infty)$, both $\boldsymbol{\tau}_{\mathbf{V}}$ and $\tau_{\mathbf{V}}$ are again of compact type.

Consider a non-empty subset $\mathbf{V} \subset \mathbf{V}(G_\infty)$. By (4.35) we obtain a commutative diagram

(6.37)
$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}_{\mathbf{V}}/\mathfrak{U}^{\mathrm{ev}}_{\mathbf{V}}} \arrow[d, "\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\mathbf{V}},\mathrm{ev}}"'] \arrow[dr, bend left, "\varphi_{\boldsymbol{\tau}_{\mathbf{V}},\mathrm{ev}}"] & & & \\
\mathbb{E}^{\mathrm{red}}_{\mathscr{U}_{\mathbf{V}}/\mathfrak{U}^{\mathrm{ev}}_{\mathbf{V}}} \arrow[r] & \mathbb{E}_{\mathscr{U}_{\mathbf{V}}/\mathfrak{U}^{\mathrm{ev}}_{\mathbf{V}}} \arrow[r] & \mathbb{F}_{\mathscr{U}_{\mathbf{V}}} \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

where the bottom sequence is a distinguished triangle, and the two arrows $\varphi_{\boldsymbol{\tau}_{\mathbf{V}},\mathrm{ev}}$ and $\varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\mathbf{V}},\mathrm{ev}}$ are the canonical and the reduced perfect obstruction theory of $\mathscr{U}_{\mathbf{V}} \to \mathfrak{U}^{\mathrm{ev}}_{\mathbf{V}}$.

Recall that $\mathscr{U}_{\mathbf{V}} = \mathscr{U}_V$ and $\mathfrak{U}^{\mathrm{ev}}_{\mathbf{V}} = \mathfrak{U}^{\mathrm{ev}}_V$ when $\mathbf{V} = \{V\}$. In this case we will replace $\mathbf{V}$ by V in Diagram (6.37) to emphasize the single vertex case.

The morphisms over $\mathscr{U}_{\mathbf{V}|\mathbf{V}^c}$

(6.38)
$$\varphi^{\mathrm{red}}_{\mathbf{V}|\mathbf{V}^c} := \varphi^{\mathrm{red}}_{\boldsymbol{\tau}_{\mathbf{V}},\mathrm{ev}} \oplus \bigoplus_{V\in\mathbf{V}^c} \varphi^{\mathrm{red}}_{\tau_V,\mathrm{ev}}, \qquad \varphi_{\mathbf{V}|\mathbf{V}^c} := \varphi_{\boldsymbol{\tau}_{\mathbf{V}},\mathrm{ev}} \oplus \bigoplus_{V\in\mathbf{V}^c} \varphi_{\tau_V,\mathrm{ev}},$$

define the reduced and canonical perfect obstruction theories of $\mathscr{U}_{\mathbf{V}|\mathbf{V}^c} \to \mathfrak{U}_{\mathbf{V}|\mathbf{V}^c}$, hence the corresponding virtual cycles $[\mathscr{U}_{\mathbf{V}^a|\mathbf{V}^c}]^{\mathrm{red}}$ and $[\mathscr{U}_{\mathbf{V}^a|\mathbf{V}^c}]^{\mathrm{vir}}$.

By (6.30), we may further pull-back (6.38) and obtain the reduced and canonical perfect obstruction theories $\varphi^{\mathrm{red}}_{\mathbf{V}^a|\mathbf{V}^c}$ and $\varphi_{\mathbf{V}^a|\mathbf{V}^c}$ of $\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c} \to \mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}$, hence the corresponding reduced and canonical virtual cycles $[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}]^{\mathrm{red}}$ and $[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}]^{\mathrm{vir}}$.

By (6.37), we have a commutative diagram over $\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}$

(6.39)
$$\begin{tikzcd}
\mathbb{T}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}} \arrow[d, "\varphi^{\mathrm{red}}_{\mathbf{V}^a|\mathbf{V}^c}"'] \arrow[dr, bend left, "\varphi_{\mathbf{V}^a|\mathbf{V}^c}"] & & & \\
\mathbb{E}^{\mathrm{red}}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}} \arrow[r] & \mathbb{E}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}} \arrow[r] & \mathbb{F}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}} \arrow[r, "{[1]}"] & {}
\end{tikzcd}$$

where the bottom is a distinguished triangle, and

(6.40)
$$\mathbb{F}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}^a|\mathbf{V}^c}} \cong \left(\mathscr{O}(\tilde{r}e_{\max,\mathbf{V}}) \oplus \bigoplus_{V\in\mathbf{V}^c} \mathscr{O}(\tilde{r}e_{\max,V})\right)[-1].$$

Recall the Cartesian squares from (6.33)

$$\begin{tikzcd}
\mathscr{U}(\boldsymbol{\tau}_\infty) \arrow[d] & \mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset} \arrow[l, "F_{\mathscr{U}}"'] \arrow[r, "\mathrm{Bl}_\infty\circ\mathrm{align}"] \arrow[d] & \prod_{V\in\mathbf{V}(G_\infty)}\mathscr{U}^{\curlyvee}_V \arrow[d] \\
\mathfrak{U}^{\mathrm{ev}}(\tau_\infty) & \mathfrak{U}^{\mathrm{ev},\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset} \arrow[l] \arrow[r] & \prod_{V\in\mathbf{V}(G_\infty)}\mathfrak{U}^{\mathrm{ev},\curlyvee}_V
\end{tikzcd} \tag{6.41}$$

The reduced virtual cycle $[\mathscr{U}(\boldsymbol{\tau}_\infty)]^{\mathrm{red}}$ can be decomposed to the vertices contributions $[\mathscr{U}_V]^{\mathrm{red}}$ as follows.

Theorem 6.12. *We have the following reduced push-forwards:*

$$F_{\mathscr{U},*}[\mathscr{U}^{\curlyvee}_V]^{\mathrm{red}} = [\mathscr{U}_V]^{\mathrm{red}}, \qquad \forall\ V\in\mathbf{V}(G_\infty). \tag{6.42}$$

$$F_{\mathscr{U},*}[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}} = [\mathscr{U}(\boldsymbol{\tau}_\infty)]^{\mathrm{red}}, \tag{6.43}$$

$$\mathrm{Bl}_{\infty,*}\,\mathrm{align}_*\left(\frac{\tilde{r}t[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}}}{-t-\psi_{\min}}\right) = \prod_{V\in\mathbf{V}(G_\infty)}\frac{\tilde{r}t[\mathscr{U}^{\curlyvee}_V]^{\mathrm{red}}}{-t-\psi_{\min,V}}. \tag{6.44}$$

Consider the composition

$$\mathrm{spl}_\infty\colon \mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset} \longrightarrow \prod_{V\in\mathbf{V}(G_\infty)}\mathscr{U}^{\curlyvee}_V \longrightarrow \prod_{V\in\mathbf{V}(G_\infty)}\mathscr{U}_V$$

Taking coefficients of the t^0-terms in (6.44) and further applying (6.42), we obtain

Corollary 6.13. $\mathrm{spl}_{\infty,*}\left((-\tilde{r})[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}}\right) = \prod_{V\in\mathbf{V}(G_\infty)}\left((-\tilde{r})[\mathscr{U}_V]^{\mathrm{red}}\right).$

6.6.2. *Proof of Theorem 6.12.* Note that the reduced perfect obstruction theory defining $[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}}$ is pulled back from $\mathscr{U}(\boldsymbol{\tau}_\infty)$ via the left Cartesian square in (6.41). Thus (6.43) follows from Lemma 6.6 and the virtual push-forward. The push-forward (6.42) can be viewed as a special case of (6.43) when $|\mathbf{V}(G_\infty)| = 1$. It remains to verify (6.44).

Consider non-empty subsets $\mathbf{V}_1 \supset \mathbf{V}_2$ of $\mathbf{V}(G_\infty)$ such that $\mathbf{V}_1\setminus\mathbf{V}_2 = \{\widetilde{V}\}$. Recall the Cartesian square from (6.31)

$$\begin{tikzcd}
\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c} \arrow[r, "\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2}"] \arrow[d] & \mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_2^a|\mathbf{V}_2^c} \arrow[d] \\
\mathfrak{U}^{\mathrm{ev},\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c} \arrow[r, "\mathfrak{align}_{\mathbf{V}_1\supset\mathbf{V}_2}"'] & \mathfrak{U}^{\mathrm{ev},\curlyvee\curlyvee}_{\mathbf{V}_2^a|\mathbf{V}_2^c}
\end{tikzcd} \tag{6.45}$$

We have the following reduced virtual push-forward.

Lemma 6.14. $\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red}} = -\tilde{r}\psi_{\min}\cdot[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_2^a|\mathbf{V}_2^c}]^{\mathrm{red}}.$

Proof. The reduced perfect obstruction theory $\varphi^{\mathrm{red}}_{\mathbf{V}_2^a|\mathbf{V}_2^c}$ pulls back to a perfect obstruction theory $\varphi^{\mathrm{red},\prime}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$ of $\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c}\to\mathfrak{U}^{\mathrm{ev},\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$, defining a virtual cycle $[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red},\prime}$. By Lemma 6.5, we have the virtual push-forward

$$\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red},\prime} = [\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}_2^a|\mathbf{V}_2^c}]^{\mathrm{red}}. \tag{6.46}$$

To compare the two virtual cycles $[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red}}$ and $[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red},\prime}$, consider the complex $\mathbb{F}'_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}$ over $\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$ given by

$$\mathbb{F}'_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}[1] = \mathrm{align}^*_{\mathbf{V}_1\supset\mathbf{V}_2}\mathbb{F}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}}[1] \cong \mathcal{O}(\tilde{r}e_{\max,\mathbf{V}_2}) \oplus \bigoplus_{V\in\mathbf{V}_2^c}\mathcal{O}(\tilde{r}e_{\max,V}).$$

By (6.20), we obtain an exact sequence of line bundles over $\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$

$$0 \longrightarrow \mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2}) \longrightarrow \mathcal{O}(\tilde{r}e_{\max,\mathbf{V}_2}) \oplus \mathcal{O}(\tilde{r}e_{\max,\widetilde{V}}) \longrightarrow \mathcal{O}(\tilde{r}e_{\max,\mathbf{V}_1}) \longrightarrow 0,$$

hence a distinsuihed triangle

$$\mathbb{F}'_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \longrightarrow \mathbb{F}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \longrightarrow \mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2}) \xrightarrow{[1]} \tag{6.47}$$

Further observe a commutative diagram over $\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}$

$$\begin{tikzcd}
\mathbb{E}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r] \arrow[d, equal] & \mathbb{F}'_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[d] \\
\mathbb{E}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r] & \mathbb{F}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}
\end{tikzcd} \tag{6.48}$$

Combining (6.39), (6.47) and (6.48), we obtain the following commutative diagram

$$\begin{tikzcd}
\mathbb{E}^{\mathrm{red},\prime}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r] \arrow[d] & \mathbb{E}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r] \arrow[d, equal] & \mathbb{F}'_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\mathbb{E}^{\mathrm{red}}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r] \arrow[d] & \mathbb{E}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r] \arrow[d] & \mathbb{F}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}} \arrow[r, "{[1]}"] \arrow[d] & {} \\
\mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2})[-1] \arrow[r] \arrow[d, "{[1]}"] & 0 \arrow[r] \arrow[d, "{[1]}"] & \mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2}) \arrow[r, "{[1]}"] \arrow[d, "{[1]}"] & {} \\
{} & {} & {} &
\end{tikzcd}$$

where both rows and columns are distinguished triangles. Taking the long exact sequence of the left column, we obtain a short exact sequence relating the obstructions of the two reduced theories:

$$0 \to H^1(\mathbb{E}^{\mathrm{red},\prime}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}) \to H^1(\mathbb{E}^{\mathrm{red}}_{\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}/\mathfrak{U}^{\mathrm{ev},\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}}) \to \mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2}) \to 0.$$

This implies the following relation

$$[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red}} = c_1(\mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2})) \cdot [\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red},\prime}. \tag{6.49}$$

Together with Lemma 6.8 and (6.46), we obtain

$$\begin{aligned}
\mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}[\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red}} &= \mathrm{align}_{\mathbf{V}_1\supset\mathbf{V}_2,*}\left(c_1(\mathcal{O}(\tilde{r}\delta_{\mathbf{V}_1\supset\mathbf{V}_2})) \cdot [\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_1^a|\mathbf{V}_1^c}]^{\mathrm{red},\prime}\right) \\
&= -\tilde{r}\psi_{\min} \cdot [\mathscr{U}^{\Upsilon\Upsilon}_{\mathbf{V}_2^a|\mathbf{V}_2^c}]^{\mathrm{red}}
\end{aligned}$$

where the second step uses [30, Thm. 4.1 (i)]. This finishes the proof. □

Set $m = |\mathbf{V}(G_\infty)|$. Now consider a sequence of non-empty subsets

$$\mathbf{V}(G_\infty) = \mathbf{V}_1 \supset \mathbf{V}_2 \supset \cdots \supset \mathbf{V}_m = \{V_m\} \tag{6.50}$$

such that $\mathbf{V}_i \setminus \mathbf{V}_{i+1} = \{V_i\}$. We have the morphism align in (6.33) decomposes to

$$\mathrm{align} = \mathrm{align}_{\mathbf{V}_1 \supset \mathbf{V}_2} \circ \mathrm{align}_{\mathbf{V}_2 \supset \mathbf{V}_3} \circ \cdots \circ \mathrm{align}_{\mathbf{V}_{m-1} \supset \mathbf{V}_m},$$

where $\mathrm{align}_{\mathbf{V}_i \supset \mathbf{V}_{i+1}} \colon \mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a_i | \mathbf{V}^c_i} \to \mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a_{i+1} | \mathbf{V}^c_{i+1}}$ is as in (6.31). Repeatedly applying Lemma 6.14 and the projection formula, we obtain

$$\mathrm{align}_*[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}} = (-\tilde{r}\psi_{\mathrm{min}})^{m-1} \cdot [\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}^a_m|\mathbf{V}^c_m}]^{\mathrm{red}} = \tilde{r}^{m-1}(-\psi_{\mathrm{min}})^{m-1} \cdot [\mathscr{U}^{\curlyvee\curlyvee}_\infty]^{\mathrm{red}}. \tag{6.51}$$

Further observe that

$$\frac{t(-\psi_{\mathrm{min}})^{m-1}}{-t-\psi_{\mathrm{min}}} = \left(t(-\psi_{\mathrm{min}})^{m-2} + t^2(-\psi_{\mathrm{min}})^{m-3} + \cdots + t^{m-1}\right) + \frac{t^m}{-t-\psi_{\mathrm{min}}}.$$

As Bl_∞ factors through (6.25), we have $\mathrm{Bl}_{\infty,*}\,\psi^k_{\mathrm{min}} = 0$ for $k \leq m-1$. We compute that

$$\begin{aligned} \mathrm{Bl}_{\infty,*}\,\mathrm{align}_*\left(\frac{\tilde{r}t[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}}}{-t-\psi_{\mathrm{min}}}\right) &= \mathrm{Bl}_{\infty,*}\left(\frac{\tilde{r}^m t(-\psi_{\mathrm{min}})^{m-1}[\mathscr{U}^{\curlyvee\curlyvee}_{\mathbf{V}(G_\infty)^a|\emptyset}]^{\mathrm{red}}}{-t-\psi_{\mathrm{min}}}\right) \\ &= \prod_{V \in \mathbf{V}(G_\infty)} \frac{\tilde{r}t[\mathscr{U}^{\curlyvee\curlyvee}_V]^{\mathrm{red}}}{-t-\psi_{\mathrm{min},V}} \end{aligned} \tag{6.52}$$

where the first equality follows from (6.51), and the second equality follows from Lemma 6.9. This finishes the proof of (6.36). □

7. APPLICATIONS

7.1. **Gromov–Witten invariants of hypersurfaces.** Let $\mathcal{Z}$ be a smooth hypersurface defined by a section s of a line bundle $\mathbf{E}_1$ on an ambient space $\mathcal{X}$. For simplicity, we will assume that $\mathcal{X}$ is a smooth projective variety, but a similar discussion applies to the case that $\mathcal{X}$ is a Deligne–Mumford stack with projective coarse moduli.

We will work in cohomology, so that we have a decomposition of the diagonal of $\mathcal{X}$

$$[\Delta] = \sum_i \phi_i \otimes \phi^i,$$

where $\{\phi_i\}$ is a homogeneous basis of $H^*(\mathcal{X};\mathbb{Q})$ and $\{\phi^i\}$ is the Poincaré dual basis.

To study the Gromov–Witten theory of $\mathcal{Z}$ via log GLSM, we consider the following special case of the setup §2.1.1:

1. $\mathcal{X}$ is the ambient projective variety;
2. $\mathbf{E} = \mathbf{E}_1$ is the line bundle defining the hypersurface;
3. we set $\mathbf{L} = \mathcal{O}_{\mathcal{X}}$;
4. we set $r = 1$ and $a = 1$.

This defines a target $\mathfrak{P}_{\mathcal{X},\mathbf{E}_1} \to \mathbf{BC}^*_\omega$.

Let β be an effective curve class of $\mathcal{X}$, and let $\alpha_1, \ldots, \alpha_n \in H^*(\mathcal{X})$. Then, by [18, Corollary 1.10], the Gromov–Witten invariants of $\mathcal{Z}$ may be computed in terms of log GLSM invariants via the formula

$$\sum_{i_*\gamma=\beta} \int_{[\mathscr{M}_{g,n}(\mathcal{X},\gamma)]^{\mathrm{vir}}} \prod_{j=1}^n \mathrm{ev}_j^*(i^*\alpha_j) = (-1)^{1-g+\int_\beta c_1(\mathbf{E}_1)} \int_{[\mathscr{R}_{g,\mathbf{1}^n}(\mathfrak{P}_{\mathcal{X},\mathbf{E}_1},\beta)]^{\mathrm{red}}} \prod_{j=1}^n \mathrm{ev}_j^*(\alpha_j), \tag{7.1}$$

where $i \colon \mathcal{Z} \to \mathcal{X}$ is the inclusion, and where $\mathbf{1}^n$ stands for n markings, each with contact order 0 and the trivial 0-sector $\mathbf{0}_{\mathfrak{P}_\mathbf{k}}$.

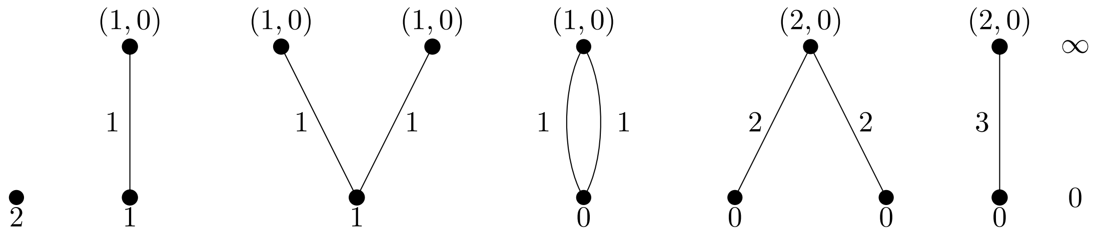

FIGURE 2. List of the partitions $\boldsymbol{\tau}_{\curlywedge}$ of $(2, \mathbf{1}^0, \beta)$ by decorated bipartite graphs, assuming for simplicity that there does not exist any $\beta \neq 0$ such that $\int_\beta \mathbf{E} \geq 2$. The number at each edge denotes the contact order. The numbers at each vertex $V \in \mathbf{V}_0$ denote $\boldsymbol{g}(V)$ and $\boldsymbol{\beta}(V)$, respectively. The single number at each vertex $V \in \mathbf{V}_\infty$ denotes $\boldsymbol{g}(V)$, with $\boldsymbol{\beta}(V)$ left arbitrary. We exclude graphs which have a vertex $V \in \mathbf{V}_0$ with a unique half-edge $h \in \mathbf{H}(V)$, and such that $\boldsymbol{c}(h) = 1$.

The tropical decomposition formula, Theorem 1.1, allows to decompose the right hand side of (7.1) as a sum over partitions $\boldsymbol{\tau}_{\curlywedge}$ of $(g, \mathbf{1}^n, \beta)$ by bipartite graphs

$$
\begin{aligned}
\int_{[\mathscr{R}_{g,\mathbf{1}^n}(\mathfrak{P}_{\mathcal{X},\mathbf{E}_1},\beta)]^{\mathrm{red}}} \prod_{j=1}^n \mathrm{ev}_j^*(\alpha_j) = \sum_{\boldsymbol{\tau}_{\curlywedge} \vdash (g,\mathbf{1}^n,\beta)} \sum_{i_E \colon E \in \mathbf{E}(G)} \frac{(-1)^{|\mathbf{V}_\infty|}}{|\operatorname{Aut}(\boldsymbol{\tau}_{\curlywedge})|} \cdot \prod_{E \in \mathbf{E}(G)} \mathbf{c}(E) \\
\cdot \prod_{V \in \mathbf{V}_\infty(G)} \int_{[\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}},\boldsymbol{\beta}(V))]^{\mathrm{red}}} \prod_{h \in \mathbf{H}(V)} \mathrm{ev}_h^*(\alpha_h) \\
\cdot \prod_{V \in \mathbf{V}_0(G)} \int_{[\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\mathfrak{P},\boldsymbol{\beta}(V))]^{\mathrm{vir}}} \prod_{h \in \mathbf{H}(V)} \mathrm{ev}_h^*(\alpha_h),
\end{aligned}
\tag{7.2}
$$

where

$$
\alpha_h = \begin{cases} \alpha_i & \text{if } h \text{ corresponds to the } i\text{th marking;} \\ \phi_{i_E} & \text{if } h \in E \cap \mathbf{H}_0; \\ \phi^{i_E} & \text{if } h \in E \cap \mathbf{H}_\infty. \end{cases}
$$

More explicitly, if $V \in \mathbf{V}_\infty(G)$, then $\vec{\varsigma}(V)$ consists of the ∞-sector with negative contact orders. When $V \in \mathbf{V}_0(G)$, then $\vec{\varsigma}(V)$ consists of copies of $\mathbf{1}$ for the legs and the ∞-sector with positive contact orders.

We further note that the stack $\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}}, \boldsymbol{\beta}(V))$ is empty unless the balancing condition

$$
\sum_{h \in \mathbf{H}(V)} \boldsymbol{c}(h) = 2g - 2 + n - \int_{\boldsymbol{\beta}(V)} c_1(\mathbf{E}).
\tag{7.3}
$$

holds, see (2.20). Hence, in (7.2), it suffices to sum only over partitions $\boldsymbol{\tau}_{\curlywedge}$ of $(g, \mathbf{1}^n, \beta)$ by decorated bipartite graphs that satisfy (7.3) for every $V \in \mathbf{V}_\infty$. In similar vein, by the stability condition (2.8), we may impose that for every $V \in \mathbf{V}_0(G)$ with $\boldsymbol{\beta}(V) = 0$ and a single half-edge $h \in \mathbf{H}(V)$, the contact order $\boldsymbol{c}(h)$ is strictly larger than 1.

In Figure 2, we list partitions of $(g, \mathbf{1}^0, \beta)$ excluding graphs that do not satisfy the balancing condition. We further exclude graphs which have a vertex $V \in \mathbf{V}_0$ with a unique half-edge $h \in \mathbf{H}(V)$, and such that $\boldsymbol{c}(h) = 1$. For any partition in Figure 2, we may obtain many additional examples of partitions by adding such vertices, see for instance Figure 3. The stability condition ensures that for a fixed value of β, there are only finitely many partitions.

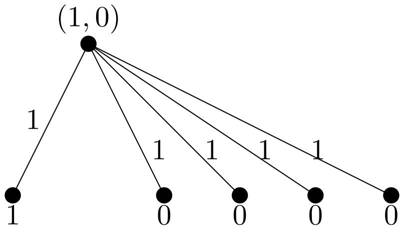

FIGURE 3. A partition $\boldsymbol{\tau}_\lambda$ of $(2, \mathbf{1}^0, \beta)$ by decorated bipartite graphs, with four vertices $V \in \mathbf{V}_0$ with a unique half-edge $h \in \mathbf{H}(V)$ with $\boldsymbol{c}(h) = 1$. We follow the same notation as in Figure 2.

All in all, by combining (7.1) and (7.2), we see that the Gromov–Witten invariants of $\mathcal{Z}$ with insertions from $\mathcal{X}$ may be effectively computed in terms of integrals of the two shapes

$$\int_{[\mathscr{R}_{g,\vec{\varsigma}}(\infty_{\mathfrak{P}},\beta)]^{\mathrm{red}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i), \qquad \int_{[\mathscr{R}_{g,\vec{\varsigma}}(\mathfrak{P},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i).$$

The first type of integral is an effective invariant in the sense of [19, §9]. This completes the proof of Corollary 1.3.

7.2. **The Calabi–Yau threefold case.** We now specialize to the case that $\mathcal{Z}$ is a Calabi–Yau hypersurface in a Fano fourfold $\mathcal{X}$. In this case, $\mathbf{E}_1 = K_{\mathcal{X}}^\vee$. For instance, if $\mathcal{Z}$ is a quintic threefold, then $\mathcal{X} = \mathbb{P}^4$ and $\mathbf{E}_1 = \mathcal{O}_{\mathbb{P}^4}(5)$. We further set $n = 0$. For simplicity, we will work in cohomology. Note that since $\mathcal{X}$ is a Fano variety, we have $H^1(\mathcal{X}) = 0$, but not necessarily $H^1(\mathcal{Z}) = 0$. Note that by the Lefschetz hyperplane principle, push-forward with respect to the inclusion yields an isomorphism $H_2(\mathcal{Z};\mathbb{Z}) \cong H_2(\mathcal{X};\mathbb{Z})$, and we can thus freely identify curves classes on $\mathcal{Z}$ and $\mathcal{X}$.

The genus $g \geq 2$ Gromov–Witten invariants of $\mathcal{Z}$ may be assembled into a generating series

$$F_g(Q) = \sum_\beta Q^\beta \deg[\mathscr{M}_{g,n}(\mathcal{Z},\beta)]^{\mathrm{vir}} \in \Lambda,$$

which is valued in the Novikov ring Λ. The series $F_g(Q)$ is called the genus g Gromov–Witten potential.

By (7.1), we have

$$F_g(Q) = (-1)^{1-g} \sum_\beta \widetilde{Q}^\beta \deg[\mathscr{R}_g(\mathfrak{P}_{\mathcal{X},\mathbf{E}_1},\beta)]^{\mathrm{red}},$$

where $\widetilde{Q}^\beta = (-1)^{\int_\beta c_1(K_{\mathcal{X}})} Q^\beta$.

We now specialize the tropical decomposition formula (7.2) to the Calabi–Yau case. Further simplification is possible due to the properties of effective invariants of [19, §9], see [19, §9.6]. First, any effective invariant with a contact order -3 or lower vanishes. Hence, it suffices to consider bipartite graphs where every edge has contact order 1 or 2. We thus get a decomposition $\mathbf{H}_0 = \mathbf{H}_1 \sqcup \mathbf{H}_2$ into half-edges with contact order 1 and contact order 2, respectively. Similarly, we may decompose $\mathbf{H}_\infty = \mathbf{H}^1 \sqcup \mathbf{H}^2$ into half-edges with contact order -1 and contact order -2, respectively. We let $\mathcal{G}_g$ be the set of partitions of (g,β) by decorated bipartite graphs for all choices of β such that the balancing condition (7.3) holds, such that every edge has contact order 1 or 2, and such that $\boldsymbol{\beta}(V) = 0$ for every $V \in \mathbf{V}_0$. As a second simplification, if $E = \{h, \hat{h}\}$ with $\boldsymbol{c}(\hat{h}) = -2$, then we need to have $\deg_{\mathbb{C}} \phi^{i_E} = 0$ in order to have a nonzero contribution to (7.2). Similarly, if $E = \{h, \hat{h}\}$ with $\boldsymbol{c}(\hat{h}) = -2$, then we need to have $\deg_{\mathbb{C}} \phi^{i_E} = 1$.

We form the following generating series corresponding to 0-vertices. For any g, n, l with $2g-2+2n+3l>0$ and $\alpha_1, \dots, \alpha_n \in H^6(\mathcal{X})$, define

$$\Omega_{g,n,l}(\alpha_1, \dots, \alpha_n) = \sum_{\beta} Q^{\beta} \int_{[\mathscr{R}_{g,\varsigma_1^n \sqcup \varsigma_2^l}(\mathfrak{P}_{\mathcal{X},\mathbf{E}_1},\beta)]^{\mathrm{vir}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i) \prod_{i=n+1}^{n+l} \mathrm{ev}_i^*(p) \in \Lambda,$$

where p is the Poincaré dual class of a point. We may define $\Omega_{0,1,0}(\alpha_1) \in \Lambda$ in the same way noting that $\mathscr{R}_{0,\varsigma_1}(\mathfrak{P}_{\mathcal{X},\mathbf{E}_1}, 0)$ is empty, and thus the series has vanishing $\beta = 0$ coefficient.

With this, we may rewrite (7.2) in a generating series form

$$\begin{aligned} F_g(Q) = (-1)^{1-g} \sum_{\boldsymbol{\tau}_{\curlywedge} \in \mathcal{G}_g} \sum_{i_E \colon E \in \mathbf{E}(G)} \frac{(-1)^{|\mathbf{V}_\infty|}}{|\operatorname{Aut}(\boldsymbol{\tau}_{\curlywedge})|} \cdot \prod_{E \in \mathbf{E}(G)} \mathbf{c}(E) \\ \cdot \prod_{V \in \mathbf{V}_\infty(G)} \widetilde{Q}^{\boldsymbol{\beta}(V)} \int_{[\mathscr{R}_{\mathbf{g}(V),\vec{\varsigma}(V)}(\infty_{\mathfrak{P}},\boldsymbol{\beta}(V))]^{\mathrm{red}}} \prod_{\hat{h} \in \mathbf{H}(V) \cap \mathbf{H}^1} \mathrm{ev}_{\hat{h}}^*(\phi^{i_{\{h,\hat{h}\}}}) \\ \cdot \prod_{V \in \mathbf{V}_0(G)} \Omega_{\boldsymbol{g}(V),|\mathbf{H}(V)\cap\mathbf{H}_1|,|\mathbf{H}(V)\cap\mathbf{H}_2|}(\phi_{i_{\{h,\hat{h}\}}} : h \in \mathbf{H}(V) \cap \mathbf{H}_1), \end{aligned} \tag{7.4}$$

where the indices i_E only index classes $\phi_i \in H^6(\mathcal{X})$ and Poincaré dual classes in $\phi^i \in H^2(\mathcal{X})$.

By the divisor equation [19, Theorem 1.16(2)], for $g \geq 2$, we may further simplify

$$\int_{[\mathscr{R}_{g,\varsigma_{-1}^n \sqcup \varsigma_{-2}^l}(\infty_{\mathfrak{P}},\beta)]^{\mathrm{red}}} \prod_{i=1}^{n} \mathrm{ev}_i^*(\alpha_i) = \left(\prod_{i=1}^{n} \int_{\beta} \alpha_i \right) \cdot c_{g,\beta}^{\mathrm{eff}}, \tag{7.5}$$

where ς_{-1} and ς_{-2} are the ∞-sectors with contact orders -1 and -2, respectively, where we have $n = 2g-2+\int_\beta K_{\mathcal{X}}$ by the balancing condition, and where the numbers

$$c_{g,\beta}^{\mathrm{eff}} = \deg[\mathscr{R}_{g,\varsigma_{-2}^{2g-2}}(\infty_{\mathfrak{P}}, \beta)]^{\mathrm{red}}$$

are called the basic effective invariants discussed in [19, §9.6]. Furthermore, when $g = 1$, the integral (7.5) vanishes unless $n = 1$, $l = 0$ and $\beta = 0$, in which case we have an explicit formula

$$\int_{[\mathscr{R}_{1,\varsigma_{-1}}(\infty_{\mathfrak{P}},0)]^{\mathrm{red}}} \mathrm{ev}_1^*(\alpha_1) = \int_{\mathcal{X} \times \mathscr{M}_{1,1}} \frac{\pi_1^*(\alpha_1) c(T_{\mathcal{X}} \boxtimes \mathbb{E}^\vee) c(\mathcal{O}_{\mathcal{X}} \boxtimes \mathbb{E}^\vee)}{c(K_{\mathcal{X}}^\vee \boxtimes \mathbb{E}^\vee)} =: c_1^{\mathrm{eff}}(\alpha_1),$$

where $\mathbb{E}$ denotes the Hodge line bundle on $\mathscr{M}_{1,1}$, and where $\pi_1 \colon \mathcal{X} \times \mathscr{M}_{1,1} \to \mathcal{X}$ is the first projection, see [19, Proposition 9.9].

We now proceed to prove the connection to mirror symmetry, Theorem 1.7. In particular, we will assume that Conjecture 1.4 holds for $\mathcal{Z}$, so that there are series $I_0(q), I_1(q) \in \Lambda'$ that define a "mirror map" isomorphism of Novikov rings:

$$\Lambda \cong \Lambda', Q^\beta = q^\beta \exp\left(\int_\beta \frac{I_1(q)}{I_0(q)} \right) \tag{7.6}$$

We also note that the series

$$K(Q) := \sum_{\beta \neq 0} Q^\beta \, \mathrm{ev}_{1,*}\left([\mathscr{R}_{0,\varsigma_1}(\mathfrak{P}, \beta)]^{\mathrm{vir}} \right) \in H^2(\mathcal{X}) \otimes \Lambda,$$

contains the same information as $\Omega_{0,1,0}$. By the assumptions of Theorem 1.7, we know that

$$K(Q) = \log(I_0(q)) - \frac{I_1(q)}{I_0(q)}, \tag{7.7}$$

and that for all g, n, l and $\alpha_1, \dots, \alpha_n$, we have

$$\Omega_{g,n,l}(\alpha_1, \dots, \alpha_n) \in (I_0(q))^{2g-2+2n+3l} \mathcal{R}$$

for a finitely generated $\mathbb{Q}$-algebra $\mathcal{R} \subseteq \Lambda'$ which contains the elements q^β for every effective curve class β. From here, our goal is to prove that

$$F_g(Q) \in (I_0(q))^{2g-2}\mathcal{R}. \tag{7.8}$$

Note that the sum in (7.4) over elements of $\mathcal{G}_g$ is infinite, see the discussion at the end of §7.1. Note further that there is a finite subset $\mathcal{G}'_g \subset \mathcal{G}_g$ of bipartite graphs that do not have any vertices $V \in \mathbf{V}_0$ such that $\boldsymbol{g}(V) = 0$ and $|\mathbf{H}(V)| = 1$, and such that for every $V \in \mathbf{V}_\infty$ such that $\boldsymbol{g}(V) = 1$, we have $|\mathbf{H}(V)| = 1$. We may use the divisor equation [19, Theorem 1.16(2)] to reduce the sum over $\mathcal{G}_g$ to a finite sum over $\mathcal{G}'_g$:

$$\begin{aligned}
F_g(Q) = (-1)^{1-g} \sum_{\boldsymbol{\tau}_\curlywedge \in \mathcal{G}'_g} \sum_{i_E \colon E \in \mathbf{E}(G)} & \frac{(-1)^{|\mathbf{V}_\infty|}}{|\operatorname{Aut}(\boldsymbol{\tau}_\curlywedge)|} \cdot \prod_{E \in \mathbf{E}(G)} \mathbf{c}(E) \\
\cdot & \prod_{\substack{V \in \mathbf{V}_\infty(G) \\ \boldsymbol{g}(V)=1, \mathbf{H}(V)=\{\hat{h}\}}} c_1^{\mathrm{eff}}(\phi^{i_{\{h,\hat{h}\}}}) \\
\cdot & \prod_{\substack{V \in \mathbf{V}_\infty(G) \\ \boldsymbol{g}(V)\geq 2}} \widetilde{Q}^{\boldsymbol{\beta}(V)} \exp\left(\int_{\boldsymbol{\beta}(V)} K(Q)\right) \left(\prod_{i=1}^n \int_\beta \alpha_i\right) \cdot c^{\mathrm{eff}}_{g,\beta} \\
\cdot & \prod_{V \in \mathbf{V}_0(G)} \Omega_{\boldsymbol{g}(V), |\mathbf{H}(V)\cap \mathbf{H}_1|, |\mathbf{H}(V)\cap\mathbf{H}_2|}(\phi_{i_{\{h,\hat{h}\}}} : h \in \mathbf{H}(V) \cap \mathbf{H}_1),
\end{aligned} \tag{7.9}$$

Now, note that by (7.7), we have

$$\widetilde{Q}^{\boldsymbol{\beta}(V)} \exp\left(\int_{\boldsymbol{\beta}(V)} K(Q)\right) = (-1)^{\int_{\boldsymbol{\beta}(V)} c_1(K_\mathcal{X})} q^{\boldsymbol{\beta}(V)} (I_0(q))^{\int_{\boldsymbol{\beta}(V)} c_1(K^\vee_\mathcal{X})}$$

Thus, to prove (7.8), it suffices to check that each summand in (7.9) has the correct power of $I_0(q)$. This follows by the computation

$$\begin{aligned}
& \sum_{\substack{V \in \mathbf{V}_\infty(G) \\ \boldsymbol{g}(V) \geq 2}} \int_{\boldsymbol{\beta}(V)} c_1(K^\vee_\mathcal{X}) + \sum_{V \in \mathbf{V}_0(G)} (2\boldsymbol{g}(V) - 2 + 2|\mathbf{H}(V) \cap \mathbf{H}_1| + 3|\mathbf{H}(V) \cap \mathbf{H}_2|) \\
= & \sum_{V \in \mathbf{V}_\infty(G)} (2\boldsymbol{g}(V) - 2 - |\mathbf{H}(V) \cap \mathbf{H}^2|) + \sum_{V \in \mathbf{V}_0(G)} (2\boldsymbol{g}(V) - 2 + 2|\mathbf{H}(V) \cap \mathbf{H}_1| + 3|\mathbf{H}(V) \cap \mathbf{H}_2|) \\
= & \sum_{V \in \mathbf{V}_\infty(G)} (2\boldsymbol{g}(V) - 2 + |\mathbf{H}(V)|) + \sum_{V \in \mathbf{V}_0(G)} (2\boldsymbol{g}(V) - 2 + |\mathbf{H}(V)|) \\
= & 2g - 2,
\end{aligned}$$

where in the first step we used the balancing condition (7.3). This completes the proof of Theorem 1.7.

APPENDIX A. BASIC NOTIONS FROM LOG GEOMETRY

A.1. **Cones and their complexes.** A *rational polyhedral cone* is a pair $\sigma = (\sigma_\mathbb{R}, N)$ where $N \cong \mathbb{Z}^n$ is a lattice and $\sigma_\mathbb{R}$ is an n-dimensional strongly convex rational polyhedral cone in $N \otimes_\mathbb{Z} \mathbb{R}$. We write $\sigma_\mathbb{Z} := \sigma_\mathbb{R} \cap N$ for the toric monoid associated to σ. Given two cones $\sigma_i = (\sigma_{i,\mathbb{R}}, N_i)$ for $i = 1, 2$ (not necessarily of the same dimension), a *morphism* $\varphi \colon \sigma_1 \to \sigma_2$ is a homomorphism of lattices $N_1 \to N_2$ such that $\varphi(\sigma_{1,\mathbb{R}}) \subset \sigma_{2,\mathbb{R}}$. We call such φ a *face morphism* if φ is injective, the image $\varphi(N_1)$ is saturated in N_2, and $\varphi(\sigma_{1,\mathbb{R}})$ is a face of $\sigma_{2,\mathbb{R}}$. The category of rational polyhedral cones is denoted by **Cones**.

We frequently view $\mathbb{R}_{\geq 0}$ as an object in **Cones** with the lattice $\mathbb{Z} \subset \mathbb{R}$. Note that the cone $\mathbb{R}_{\geq 0}$ has exactly two faces 0 and $\mathbb{R}_{\geq 0}$. With these conventions, for any $\sigma \in$

Cones its *dual* $\sigma^\vee := \mathrm{Hom}_{\mathbf{Cones}}(\sigma, \mathbb{R}_{\geq 0})$ is again a cone in **Cones**, with the monoid $\sigma^\vee_{\mathbb{Z}} = \mathrm{Hom}(\sigma_{\mathbb{Z}}, \mathbb{N})$.

For later use, note that for any $\sigma \in$ **Cones** its monoid $\sigma_{\mathbb{Z}}$ is a partially ordered set such that $m_1 \preccurlyeq_\sigma m_2$ iff $m_2 - m_1 \in \sigma_{\mathbb{Z}}$, for any $m_1, m_2 \in \sigma_{\mathbb{Z}}$. We may write $m_1 \preccurlyeq m_2$ for simplicity when there is no confusion of the cone σ.

A *generalized cone complex* is a topological space with a presentation as the colimit of a finite diagram in the category **Cones** with all morphisms being face morphisms. For example, a cone $\sigma \in$ **Cones** is naturally a cone complex of all its faces. For a generalized cone complex Σ, we write $\sigma \in \Sigma$ if σ is a cone in the diagram defining Σ, and write $|\Sigma|$ for the underlying topological space. A *morphism* of generalized cone complexes $f \colon \Sigma \to \Sigma'$ is a continuous map $|\Sigma| \to |\Sigma'|$ such that for each $\sigma \in \Sigma$, the induced map $\sigma_{\mathbb{R}} \to |\Sigma'|$ factors through a morphism $\sigma \to \sigma' \in \Sigma'$.

A.2. **Deligne–Faltings log structures of rank one.**

In this paper, the notion of Deligne–Faltings log structures will play a crucial role. We recall them below.

A.2.1. *The stack of Deligne–Faltings log structures of rank one.* Recall that a log stack X is *Deligne-Faltings type of rank* 1 (or simply *DF1*) if there is a morphism $\mathbb{N} \to \Gamma(X, \overline{\mathcal{M}}_X)$ which locally lifts to a chart of $\mathcal{M}_X$.

The universal DF1 target is the log stack $\mathcal{A}$ with $\underline{\mathcal{A}} = [\mathbb{A}^1/\mathbb{G}_m]$ and its divisorial log structure given by the origin $[0/\mathbb{G}_m]$, denoted by $\infty_{\mathcal{A}}$ with its log structure pulled back from $\mathcal{A}$. Note that the natural morphism $\mathbb{N} \cong \Gamma(\mathcal{A}, \overline{\mathcal{M}}_{\mathcal{A}}) \to \overline{\mathcal{M}}_{\mathcal{A}}$ smooth-locally lifts to a chart of $\mathcal{M}_{\mathcal{A}}$. A log stack X is of DF1 iff there is a natural strict morphism $X \to \mathcal{A}$. In particular, if $\mathcal{M}_X$ is the divisorial log structure associated to a smooth divisor $D \subset X$, then the natural strict morphism $\phi \colon X \to \mathcal{A}$ is defined such that the $\phi^{-1}(\infty_{\mathcal{A}}) = D$.

Let X be a DF1 log stack with the natural morphism $X \to \mathcal{A}$. Consider the generator $1_{\mathcal{A}} \in \mathbb{N}_{\mathcal{A}}$ with image $1_X \in \Gamma(X, \overline{\mathcal{M}}_X)$. The pre-image $\mathcal{T} := 1_X \times_{\overline{\mathcal{M}}_X} \mathcal{M}_X \subset \mathcal{M}_X$ is an $\mathcal{O}^*$-torsor. The restriction of the structural morphism $\alpha \colon \mathcal{M}_X \to \mathcal{O}_X$ to $\mathcal{T}$ extends to a morphism $s^\vee$ of line bundles on X:

$$\mathcal{T} \subset \mathcal{O}(-1_X) \xrightarrow{s^\vee} \mathcal{O}_X, \tag{A.1}$$

where $s^\vee$ vanishes precisely along the pre-image of $\infty_{\mathcal{A}} \subset \mathcal{A}$. Conversely, by [27, Complement 1] the dual $s \in \Gamma(X, \mathcal{O}(1_X) := \mathcal{O}(-1_X)^\vee)$ of $s^\vee$ determines $\mathcal{M}_X$.

A.2.2. *Line bundles associated to global sections of characteristic sheaves.* Let $\mathfrak{M}$ be a log stack and $e \in \Gamma(\mathfrak{M}, \overline{\mathcal{M}}_{\mathfrak{M}})$ be a global section. Similar to (A.1), the section e induces

$$\mathcal{M}_{\mathfrak{M}} \times_{\overline{\mathcal{M}}_{\mathfrak{M}}} \{e\} \subset \mathcal{O}_{\mathfrak{M}}(-e) \xrightarrow{\mathbf{s}_e^\vee} \mathcal{O}_{\mathfrak{M}}, \tag{A.2}$$

extending the $\mathcal{O}^*$-torsor $\mathcal{M}_{\mathfrak{M}} \times_{\overline{\mathcal{M}}_{\mathfrak{M}}} \{e\}$ to a line bundle $\mathcal{O}_{\mathfrak{M}}(-e)$ with a morphism $\mathbf{s}_e^\vee$.

Denote by $\mathcal{O}_{\mathfrak{M}}(e) := \mathcal{O}_{\mathfrak{M}}(-e)^\vee$, and the section $\mathbf{s}_e := \mathbf{s}_e^\vee \otimes \in \Gamma(\mathfrak{M}, \mathcal{O}_{\mathfrak{M}}(e))$ By [27, Complement 1], the pair $(\mathcal{O}_{\mathfrak{M}}(e), \mathbf{s}_e)$ defines a rank one Deligne–Faltings log structure, denoted by $\mathcal{M}$. Consequently, we obtain a strict morphism

$$(\underline{\mathfrak{M}}, \mathcal{M}) \xrightarrow{(\mathcal{O}_{\mathfrak{M}}(e), \mathbf{s}_e)} \mathcal{A}. \tag{A.3}$$

with $\mathcal{O}_{\mathfrak{M}}(e) \cong \mathcal{O}(\infty_{\mathcal{A}})|_{\mathfrak{M}}$. The vanishing locus of $\mathbf{s}_e$ is

$$\mathrm{Div}(e) = \underline{\mathfrak{M}} \times_{\mathcal{A}} \infty_{\mathcal{A}} \subset \underline{\mathfrak{M}} \tag{A.4}$$

which is also set-theoretically the non-vanishing locus of e. Locally on a strict smooth chart $U \to \mathfrak{M}$, the locus $\mathrm{Div}(e)$ can be described as follows. Choose a lift $\tilde{e}_U \in \Gamma(U, \mathcal{M}_U)$ of $e|_U$. Then $\mathrm{Div}(e) \times_{\underline{\mathfrak{M}}} \underline{U}$ is the substack defined by the vanishing of $\alpha(\tilde{e}_U)$. Note that $\mathrm{Div}(e) \times_{\underline{\mathfrak{M}}} \underline{U}$ hence $\mathrm{Div}(e)$ does not depend on the choices of local liftings $\tilde{e}_U$. In case $e|_{\mathfrak{M}^\circ} = 0$ over an open dense substack $\mathfrak{M}^\circ \subset \mathfrak{M}$, the locus $\mathrm{Div}(e) \subset \underline{\mathfrak{M}}$ is a *Cartier divisor.*

When the section $\mathbf{s}_e$ vanishes identically, we write

$$(\underline{\mathfrak{M}}, \mathcal{M}) \xrightarrow{(\mathcal{O}_{\mathfrak{M}}(e),0)} \mathcal{A} \qquad \text{or simply} \qquad (\underline{\mathfrak{M}}, \mathcal{M}) \xrightarrow{\mathcal{O}_{\mathfrak{M}}(e)} \infty_{\mathcal{A}} \tag{A.5}$$

as the left morphism factors through the strict closed substack $\infty_{\mathcal{A}} \subset \mathcal{A}$.

Further observe that $\mathcal{M}$ is the sub-log structure of $\mathcal{M}_{\mathfrak{M}}$ generated by $\mathcal{M}_{\mathfrak{M}} \times_{\overline{\mathcal{M}}_{\mathfrak{M}}} \{e\} \subset \mathcal{M}$. Hence we obtain the composition

$$\mathfrak{M} \longrightarrow (\underline{\mathfrak{M}}, \mathcal{M}) \xrightarrow{(\mathcal{O}_{\mathfrak{M}}(e),\mathbf{s}_e)} \mathcal{A}. \tag{A.6}$$

with the left arrow is given by the inclusion $\mathcal{M} \subset \mathcal{M}_{\mathfrak{M}}$.

Appendix B. Punctured maps and their tropicalizations

In this appendix, we review the theory of *punctured logarithmic maps*, or simply *punctured maps* of Abramovich-Chen-Gross-Siebert [3], which provides the logarithmic framework needed in this paper. Most of the basic results will be stated in a setting with domains (not necessarily connected) orbifold curves and targets with Deligne–Faltings log structures of rank 1 — the setting that is necessary for our applications in log GLSM. Since almost all results follow from identical proofs as in [3], to avoid unnecessary repetition, we will refer the readers to the relevant parts in [3] and only point out the differences.

B.1. **Pre-stable curves with log structures.**

B.1.1. *Pre-stable curves.* In this paper, a *pre-stable curve* over a scheme $\underline{S}$ means a *twisted n-pointed curve* in the sense of [6] and consists of the following data

$$(\underline{C} \to \underline{C}^c \to \underline{S}, \{\underline{p_i}\}_{i=1}^n) \tag{B.1}$$

where

1. $\underline{C}$ is a proper Deligne–Mumford stack, and is étale locally a nodal curve over $\underline{S}$.
2. $\underline{p_i} \subset \underline{C}$ are disjoint closed substacks in the smooth locus of $\underline{C} \to \underline{S}$.
3. $\underline{p_i} \to \underline{S}$ are étale gerbes banded by the multiplicative group μ_{m_i} for some positive integer m_i.
4. the morphism $\underline{C} \to \underline{C}^c$ is the coarse moduli morphism.
5. along each nodal locus of $\underline{C} \to \underline{S}$, the group action of μ_{m_i} is balanced.
6. $\underline{C} \to \underline{C}^c$ is an isomorphism over $\underline{C}_{gen}$, where $\underline{C}_{gen}$ is the complement of the markings and the stacky locus of $\underline{C} \to \underline{S}$.

In the above definition, we allow $\underline{C}$ hence $\underline{C}^c$ to be disconnected. Given a twisted curve as above, by [6, 4.11] the coarse space $\underline{C}^c \to \underline{S}$ is a family of (possibly disconnected) n-pointed usual pre-stable curves over $\underline{S}$ with the markings determined by the images of $\{\underline{p_i}\}$. The *genus* of the twisted curve $\underline{C}$ is defined as a tuple of non-negative integers, with each integer specifying the arithmetic genus of a connected component of $\underline{C}^c$.

When there is no danger of confusion, we will simply write $\underline{C} \to \underline{S}$, and the terminologies twisted curves and pre-stable curves are interchangeable in this paper.

B.1.2. *Log curves.* An *n-pointed log curve* over a fs log scheme S in the sense of [34] consists of a pair

$$(\pi\colon C \to S, \{\underline{p_i}\}_{i=1}^n), \tag{B.2}$$

such that

1. The underlying data $(\underline{C} \to \underline{C}^c \to \underline{S}, \{\underline{p_i}\}_{i=1}^n)$ is a pre-stable curve as in (B.1).
2. π is a proper, logarithmically smooth and integral morphism of fine and saturated logarithmic stacks.
3. If $\underline{U} \subset \underline{C}$ is the non-singular locus of $\underline{\pi}$, then $\overline{\mathcal{M}}_C|_{\underline{U}} \cong \pi^*\overline{\mathcal{M}}_S \oplus \bigoplus_{i=1}^n \mathbb{N}_{\underline{p_i}}$ where $\mathbb{N}_{\underline{p_i}}$ is the constant sheaf over $\underline{p_i}$ with fiber $\mathbb{N}$.

For simplicity, we may refer to $\pi\colon C \to S$ as a log curve when there is no danger of confusion. The *pull-back* of a log curve $\pi\colon C \to S$ along an arbitrary morphism of fs log schemes $T \to S$ is the log curve $\pi_T\colon C_T := C \times_S T \to T$ where the fiber product is taken in the fs category.

For a log curve (B.2), its *log cotangent bundle* is $\omega^{\log}_{C/S} \cong \omega_{\underline{C}/\underline{S}}(\sum_i \underline{p_i})$ where $\omega_{\underline{C}/\underline{S}}$ is the relative dualizing line bundle of the underlying $\underline{C} \to \underline{S}$.

B.1.3. *Characteristic sheaves of log curves.* For the reader's convenience, we recall the local combinatorial structure of log curves following [20, §2.1.5]. Let $C \to S$ be a family of log curves, $\overline{\mathcal{M}}_C$ and $\overline{\mathcal{M}}_S$ the corresponding characteristic sheaves, and $p \in \underline{C}$ a geometric point. There are three cases.

If p is a smooth non-marked point, then we have the fiber $\overline{\mathcal{M}}_C|_p \cong \overline{\mathcal{M}}_S$.

If p is a marked point, then we have the fiber $\overline{\mathcal{M}}_C|_p \cong \overline{\mathcal{M}}_S \oplus \mathbb{N}$. Indeed, let $x \in \mathcal{M}_C|_p$ be pre-image of the element $(0,1) \in \overline{\mathcal{M}}_S \oplus \mathbb{N}$. Then the image $\alpha(x)$ via the structural morphism $\alpha\colon \mathcal{M}_C \to \mathcal{O}_C$ is a local coordinate, whose vanishing defines the marking p.

If p is a node, then we have the fiber $\overline{\mathcal{M}}_C|_p \cong \overline{\mathcal{M}}_S \oplus_{\mathbb{N}} \mathbb{N}^2$ given by the diagonal $\mathbb{N} \to \mathbb{N}^2, 1 \mapsto (1,1)$ and an inclusion $\mathbb{N} \to \overline{\mathcal{M}}_S, 1 \mapsto \ell$. Similar to the case of markings, the two generators $(1,0),(0,1)$ of the factor $\mathbb{N}^2$ correspond to the local coordinates of the two components of the node. Thus, we have the relation

$$(1,0) + (0,1) = \ell, \qquad \text{in } \overline{\mathcal{M}}_C. \tag{B.3}$$

The element ℓ is called the *smoothing parameter* of the node p. Indeed, shrinking S, we may choose any chart $\beta\colon \overline{\mathcal{M}}_{S,s} \to \mathcal{M}_S$ for s the image of p via the projection $C \to S$. Then the closed subscheme $V \subset S$ defined by $\alpha \circ \beta(\ell) = 0$ is the locus where the connected component of nodes containing p persists.

B.1.4. *Punctured curves.* Given a log curve $C \to S$, there is a natural splitting of the log structure $\mathcal{M}_C = \mathcal{N}_C \oplus_{\mathcal{O}^*} \mathcal{P}_C$, where $\mathcal{P}_C \subset \mathcal{M}_C$ is the divisorial log structure given by the markings. In particular, the characteristic monoid is given by $\overline{\mathcal{P}}_C = \bigoplus_{i=1}^n \mathbb{N}_{p_i}$.

A *punctured curve* over a fine and saturated log scheme S in the sense of [3, §2.1.4] consists of

$$(C^\circ \xrightarrow{\mathrm{p}} C \xrightarrow{\pi} S, \{\underline{p_i}\}_{i=1}^n), \tag{B.4}$$

where $(C \xrightarrow{\pi} S, \{\underline{p_i}\}_{i=1}^n)$ is a log curve as in (B.2), and $\mathrm{p}\colon C^\circ \to C$ is a morphism of fine log stacks ($\mathcal{M}_{C^\circ}$ is not necessarily saturated) satisfying

1. The underlying morphism $\underline{\mathrm{p}}\colon \underline{C^\circ} \to \underline{C}$ is an isomorphism.
2. $\mathrm{p}^\flat\colon \mathcal{M}_C \to \mathcal{M}_{C^\circ}$ is an isomorphism away from the collection of markings.
3. $\mathrm{p}^\flat$ induces a sequence of inclusions of sheaves of fine monoids

$$\mathcal{M}_C \overset{\mathrm{p}^\flat}{\hookrightarrow} \mathcal{M}_{C^\circ} \subset \mathcal{N}_C \oplus_{\mathcal{O}^*} \mathcal{P}^{gp}. \tag{B.5}$$

4. For any marking $p \subset \underline{\mathcal{C}}$ and any element $s \in \mathcal{M}_{C^\circ,p}$ be such that $s \notin \mathcal{M}_{C,p}$, we require the vanishing $\alpha_{C^\circ}(s) = 0$.

The morphism p is called the *puncturing of $\mathcal{C}$ along $\mathcal{P}$ (or along the markings)*. Markings of a punctured curve are called *punctured markings*, or simply *punctures*. If $\mathrm{p}|_{\underline{p_i}}$ is an isomorphism, we say that the puncturing is *trivial* along $\underline{p_i}$, and $\underline{p_i}$ is a *log marking*. For a marking $\underline{p_i}$, denote by $p_i \subset C^\circ$ the corresponding strict closed substack. For simplicity, we may refer to $(C^\circ \to S, \{p_i\})$ or even $C^\circ \to S$ as a punctured curve when there is no danger of confusion.

Remark B.1. Consider a puncture p and an element $s \in \mathcal{M}_{C^\circ,p}$ satisfying $s \notin \mathcal{M}_{C^\circ}$ as in (4) above. We may write s in the form

$$s = (m, y) \in \mathcal{N}_{C,p} \oplus_{\mathcal{O}^*} \mathcal{P}^{gp}_p, \tag{B.6}$$

By [3, Remark 2.2], we necessarily have the vanishing $\alpha_C(m) = 0$. This is a local obstruction for deforming punctured curves, and is captured by the idealized log structure in §B.1.5 below.

The *pull-back* of a punctured curve (B.4) along a morphism of fs log schemes $T \to S$ consists of the following data

$$(C_T^\circ \xrightarrow{\mathrm{p}} C_T \xrightarrow{\pi} T, \{p_{i,T}\}_{i=1}^n), \tag{B.7}$$

where $(C_T \xrightarrow{\pi} T, \{p_{i,T}\}_{i=1}^n)$ is the pull-back of the corresponding log curve, and $C_T^\circ = C_T \times_C C^\circ$ with the fiber product taken in the fine category.

It is a non-trivial fact that the pull-back of punctured curves defined as above is again a punctured curve, see [3, Proposition 2.7].

B.1.5. *Puncturing log-ideals.* Let S be a log scheme or a log stack. A *log-ideal* over S is a sheaf of monoid ideals $\mathcal{K} \subset \mathcal{M}_S$. The sheaf of monoid ideals $\mathcal{K}$ is *coherent* if it is locally finitely generated, see [31, II.2.6.1]. In this article, we will assume that all sheaves of monoid ideals are coherent. The pair $(S, \mathcal{K})$ is called an *idealized log scheme (or stack)* if $\alpha_S(\mathcal{K}) = 0$.

For a punctured curve $\pi\colon C^\circ \to S$, it admits a natural idealized structures from its punctures as follows. For each puncture $p \subset C^\circ$, consider the composition

$$\upsilon_p\colon \mathcal{M}_{C^\circ}|_p \to \overline{\mathcal{M}}_S \oplus \mathbb{Z} \to \mathbb{Z}$$

where the first arrow is given by (B.5) and the local structure at the marking §B.1.3, and the second arrow is the projection. Denote by $\mathcal{K}_p \subset \mathcal{M}_{C^\circ}|_p$ the sheaf of monoid ideals generated by $\upsilon_p^{-1}(\mathbb{Z}_{<0})$. The *puncturing log-ideal* $\mathcal{K}_S^\circ \subset \mathcal{M}_S$ is the monoid ideal generated by $\bigcup_p (\pi^\flat)^{-1}(\mathcal{K}_p) \subset \mathcal{M}_S$ where p runs through all punctures, and $\pi\colon C^\circ \to S$ is the projection. It is shown in [3, §2.5.2] that $\mathcal{K}_S^\circ$ is a coherent log ideal satisfying $\alpha(\mathcal{K}_S^\circ) = 0$. In particular, the base $(S, \mathcal{K}_S^\circ)$ is naturally an idealized log scheme.

Indeed, the pull-back log-ideal $\pi^\bullet \mathcal{K}_S^\circ$ is generated by all such $m \in \mathcal{N}_{C,p}$ as in (B.6). Thus the idealized structure $\alpha(\mathcal{K}_S^\circ) = 0$ captures precisely the vanishing in Remark B.1.

It is further proved in [3, Prop. 2.51] that the puncturing log-ideals are well-behaved under pull-backs of punctured curves. Suppose $C_T^\circ \to T$ is a punctured curve obtained by pulling back pulled back $C^\circ \to S$ along a morphism $h\colon T \to S$, then the puncturing log-ideal over T is the pull-back $\mathcal{K}_T^\circ = h^\bullet(\mathcal{K}_S^\circ)$.

B.1.6. *Graphs and their contractions.* The combinatorial structures of pre-stable curves can be conveniently encoded in their dual graphs. We recall the notion of graphs following [9, §3.1], and introduce extensions needed in this paper.

A *graph* G consists of the following data:

1. A finite set $\mathbf{V}(G) \sqcup \mathbf{H}(G)$, where $\mathbf{V}(G)$ is the set of vertices of G, and $\mathbf{H}(G)$ is the set of *half-edges* (or sometimes called flags) of G.
2. A *root map* $\nu_G\colon \mathbf{V}(G) \sqcup \mathbf{H}(G) \longrightarrow \mathbf{V}(G)$ which is an idempotent.
3. An *involution* $\iota_G\colon \mathbf{V}(G) \sqcup \mathbf{H}(G) \longrightarrow \mathbf{V}(G) \sqcup \mathbf{H}(G)$ whose fixed point set contains $\mathbf{V}(G)$.

The vertex $\nu_G(h) \in \mathbf{V}(G)$ stands for the vertex the half-edge $h \in \mathbf{H}(G)$ emanates from. A half-edge h is called a *leg* if $\iota_G(h) = h$. Otherwise, a pair $E = \{h, \iota_G(h)\}$ of distinct half-edges is called an *edge*. An edge $E = \{h, \hat{h}\}$ is called a *loop* if $\nu_G(h) = \nu_G(\hat{h})$. We will denote by $\mathbf{L}(G)$ and $\mathbf{E}(G)$ the sets of legs and edges respectively. For each $V \in \mathbf{V}(G)$, denote by $\mathbf{H}(V) = \{h \in \mathbf{H}(G) \mid \nu_G(h) = V\}$ the set of half-edges incident to V.

A graph G can be geometrically realized as a 1-complex in a natural way such that each edge $E = \{h, \hat{h}\}$ joins the vertices $\nu_G(h), \nu_G(\hat{h})$, and each leg $L \in \mathbf{L}(G)$ is a half-edge emanating from the vertex $\nu_G(L)$. A graph G is *connected* if its geometric realization is connected, or equivalently any pair of vertices of G is connected by a path of edges.

A *morphism* of graphs $\phi\colon G' \to G$ is a map of sets $\phi_{\mathbf{V}}\colon \mathbf{V}(G') \sqcup \mathbf{H}(G') \to \mathbf{V}(G) \sqcup \mathbf{H}(G)$ such that

1. $\phi \circ \nu_{G'} = \nu_G \circ \phi$ and $\phi \circ \iota_{G'} = \iota_G \circ \phi$;
2. ϕ induces a bijection of legs $\mathbf{L}(\phi)\colon \mathbf{L}(G') \to \mathbf{L}(G)$;
3. For each half-edge $h \in \mathbf{H}(G)$, the preimage $\phi^{-1}(h)$ consists of a unique element in $\mathbf{H}(G')$;
4. For each vertex $V \in \mathbf{V}(G)$, the preimage $\phi^{-1}(V)$ is a connected subgraph of G'.

A morphism ϕ is called a *contraction* if it is surjective. Such a contraction restricts to a surjection of the set of vertices $\mathbf{V}(\phi)\colon \mathbf{V}(G') \to \mathbf{V}(G)$, and induces an injection of the set of edges $\mathbf{E}(\phi)\colon \mathbf{E}(G) \to \mathbf{E}(G')$.

B.1.7. *Decorated graphs.* For a graph G, we further introduce the following data.

1. A *leg labeling* of G is a bijection of sets $\mathbf{m}\colon \{1, 2, \cdots, k\} \to \mathbf{L}(G)$.
2. A *genus decoration* of G is a function $\mathbf{g}\colon \mathbf{V}(G) \to \mathbb{N}$.
3. A *degree decoration* of G is a map $\mathbf{deg}\colon \mathbf{H}(G) \to \{\frac{1}{m} \mid m \in \mathbb{N} \setminus \{0\}\}$ satisfying $\mathbf{deg} \circ \iota_G = \mathbf{deg}$. Thus for an edge $E = \{h, \hat{h}\}$, we may define its degree to be $\mathbf{deg}(E) = \mathbf{deg}(h) = \mathbf{deg}(\hat{h})$.

In this paper, a triple $(G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ will be called a *decorated graph.* The *genus* of a connected decorated graph $(G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ is

$$\mathbf{g}(G) = h^1(G) + \sum_{V \in \mathbf{V}(G)} \mathbf{g}(V). \tag{B.8}$$

A *morphism* $\phi\colon (G', \mathbf{g}', \mathbf{deg}', \mathbf{m}') \longrightarrow (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ of decorated graphs is a morphism of the underlying graphs $\phi\colon G' \to G$ which preserves the leg labelings, and satisfies additional compatibilities

$$\begin{aligned} \mathbf{g}(V) &= h^1(\phi^{-1}(V)) + \sum_{V' \in \mathbf{V}(\phi)^{-1}(V)} \mathbf{g}'(V'), && \text{for any } V \in \mathbf{V}(G), \\ \mathbf{deg}'(L) &= \mathbf{deg}(\mathbf{L}(\phi)(L)), && \text{for any } L \in \mathbf{L}(G'), \\ \mathbf{deg}(E) &= \mathbf{deg}'(\mathbf{E}(\phi)(E)), && \text{for any } E \in \mathbf{E}(G). \end{aligned}$$

A *contraction* of a decorated graph is a morphism of decorated graphs which is a contraction of the underlying graphs.

B.1.8. *Curves marked by decorated graphs.* Let $\underline{C}$ be a (possibly disconnected) pre-stable curve over a geometric point. Let G' be its dual graph, whose set of vertices $\mathbf{V}(G')$ consists of the irreducible components of $\underline{C}$, and whose set of legs $\mathbf{L}(G)$ corresponds to the marked points. For each irreducible component $Z_V \subset \underline{C}$ corresponding to a vertex V, let $\widetilde{Z}_V \to Z_V$ be the normalization. The set of half-edges $\mathbf{H}(V)$ incident to V is given by the pre-image of the special points in $\widetilde{Z}_V$. In particular, the set of edges $\mathbf{E}(G')$ corresponds to nodes.

The dual graph G' is naturally equipped with the genus decoration $\mathbf{g}'$ and the degree decoration $\mathbf{deg}'$ defined as follows. For each V, its genus $\mathbf{g}'(V)$ is the genus of the corresponding irreducible component $Z_V \to \underline{C}$. Furthermore $\mathbf{deg}'(h) = \frac{1}{r_h}$ if $Z_h \cong B\mu_{r_h} \hookrightarrow \underline{C}$ is the node or marked point given by $h \in \mathbf{H}(G')$.

A *marking* of $\underline{C}$ by $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ is a contraction $\phi\colon (G', \mathbf{g}', \mathbf{deg}', \mathbf{m}') \to \mathbf{G}$ of decorated graphs. A log or a punctured curve is said to be *marked by* $\mathbf{G}$ if the corresponding underlying curve is marked by $\mathbf{G}$. Pre-stable curves marked by $\mathbf{G}$ form an algebraic stack, denoted by $\mathfrak{M}(\mathbf{G})$. By [34], the stack $\mathfrak{M}(\mathbf{G})$ is a log stack with a canonical log structure $\mathcal{M}_{\mathfrak{M}(\mathbf{G})}$ and a universal log curve $\mathfrak{C}(\mathbf{G}) \to \mathfrak{M}(\mathbf{G})$ satisfying the following universal property.

Let $C \to S$ be any log curve marked by $\mathbf{G}$. Then there exists a unique morphism of log stacks $S \to \mathfrak{M}(\mathbf{G})$ such that $C \to S$ is the pull-back of the universal family $\mathfrak{C}(\mathbf{G}) \to \mathfrak{M}(\mathbf{G})$. Thus $\mathfrak{M}(\mathbf{G})$ represents the category of log curves marked by $\mathbf{G}$.

Notation B.2. Let $C^\circ \to S$ be a punctured curve marked by $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$. For a half-edge $h \in \mathbf{H}(G)$, we denote by $p_h \subset C^\circ$ the strict substack corresponding to h as follows. If $h \in \mathbf{L}(G)$ is a leg, then p_h is the corresponding marking.

If $E = \{h, \hat{h}\} \in \mathbf{E}(G)$ forms an edge, then consider the strict morphism $\widetilde{C^\circ} \to C^\circ$ with its underlying given by the partial normalization along the node $\underline{p_E} \subset C^\circ$ corresponding to E. The pre-image of p_E has two components $p_h \sqcup p_{\hat{h}} \subset \widetilde{C^\circ}$ with $p_h \subset Z_{\nu_G(h)}$ and $p_{\hat{h}} \subset Z_{\nu_G(\hat{h})}$. Note that there is an isomorphism of log stacks over S

$$\iota_h \colon p_h \to p_{\hat{h}}, \tag{B.9}$$

whose underlying is inverting the band of gerbes over $\underline{S}$.

B.1.9. *Nodal log-ideals.* As in [3, 3.1.2], the stack $\mathfrak{M}(\mathbf{G})$ admits a natural log-ideal $\mathcal{K}^n_{\mathfrak{M}(\mathbf{G})} \subset \mathcal{M}_{\mathfrak{M}(\mathbf{G})}$, called the *nodal log-ideal* defined as follows. For each geometric point $[C \to w] \in \mathfrak{M}(\mathbf{G})$ with the dual graph G', recall that $\overline{\mathcal{M}}_{\mathfrak{M}(\mathbf{G})}|_w \cong \mathbb{N}^{|\mathbf{E}(G')|}$ with generators corresponding to the smoothing parameters of nodes, see §B.1.3. Define the *nodal monoid ideal*

$$\overline{\mathcal{K}}^n_{\mathfrak{M}(\mathbf{G})}|_w = \mathbb{N}^{\mathbf{E}(\phi)(\mathbf{E}(G))} \setminus \{0\} \subset \overline{\mathcal{M}}_{\mathfrak{M}(\mathbf{G})}|_w \tag{B.10}$$

where ϕ is the $\mathbf{G}$-marking. This fiberwise construction glues to a sheaf of monoid ideals $\overline{\mathcal{K}}^n_{\mathfrak{M}(\mathbf{G})} \subset \overline{\mathcal{M}}_{\mathfrak{M}(\mathbf{G})}$, hence a log-ideal $\mathcal{K}^n_{\mathfrak{M}(\mathbf{G})} = \mathcal{M}_{\mathfrak{M}(\mathbf{G})} \times_{\overline{\mathcal{M}}_{\mathfrak{M}(\mathbf{G})}} \overline{\mathcal{K}}^n_{\mathfrak{M}(\mathbf{G})}$, called the *nodal log-ideal.* Since the nodes labeled by edges of G are not allowed to be smoothed out over $\mathfrak{M}(\mathbf{G})$, this implies $\alpha(\mathcal{K}^n_{\mathfrak{M}(\mathbf{G})}) = 0$. Thus the pair $(\mathfrak{M}(\mathbf{G}), \mathcal{K}^{\mathbf{n}}_{\mathfrak{M}(\mathbf{G})})$ is an idealized log stack.

Let $G = \sqcup_i G_i$ be the decomposition into connected components. Let $\mathbf{G}_i$ be the decorated graph obtained by restricting decorations of $\mathbf{G}$ to the component G_i. We observe that

$$\mathfrak{M}(\mathbf{G}) = \prod_i \mathfrak{M}(\mathbf{G}_i)$$

as a product of fs log stacks.

B.1.10. *Moduli of punctured curves.* Consider the stack $\mathfrak{M}(\mathbf{G})$ of log curves marked by $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ as in §B.1.8. Let $\breve{\mathfrak{M}}(\mathbf{G})$ be the fibered category over schemes parameterizing punctured curves marked by $\mathbf{G}$, see [3, Definition 3.2]. There is a tautological commutative diagram

$$\begin{tikzcd} \breve{\mathfrak{M}}(\mathbf{G}) \arrow[rr, "\breve{F}"] \arrow[dr, "\mathrm{Log}_{\breve{F}}"'] & & \mathfrak{M}(\mathbf{G}) \\ & \mathrm{Log}_{\mathfrak{M}(\mathbf{G})} \arrow[ur] & \end{tikzcd} \tag{B.11}$$

where $\breve{F}$ sends a punctured curve to the corresponding log curves, and $\mathrm{Log}_{\mathfrak{M}(\mathbf{G})}$ is Olsson's log stack [32] parameterizing log structures over $\mathfrak{M}(\mathbf{G})$. The morphism $\mathrm{Log}_{\breve{F}}$ is strict, locally of finite type, quasi-separated, representable, and unramified.

For pre-stable curves with no orbifold structures at nodes and markings, these properties of $\mathrm{Log}_{\breve{F}}$ are established in [3, Proposition 3.3]. In this case, the stack $\breve{\mathfrak{M}}(\mathbf{G})$ is constructed locally over $\mathfrak{M}(\mathbf{G})$ by taking into account all possible puncturings along markings. Since twisted curves admit étale local covers by schemes, the same proof can be applied to the case of pre-stable curves with orbifold structures as in this paper. In particular, $\breve{\mathfrak{M}}(\mathbf{G})$ is a log algebraic stack.

Denote by $\mathcal{K}^{\circ}_{\breve{\mathfrak{M}}(\mathbf{G})} \subset \mathcal{M}_{\breve{\mathfrak{M}}(\mathbf{G})}$ the puncturing log-ideal as in §B.1.5, and consider the nodal log-ideal $\mathcal{K}^{n}_{\mathfrak{M}(\mathbf{G})} \subset \mathcal{M}_{\mathfrak{M}(\mathbf{G})}$ as in §B.1.9. Define the *canonical log-ideal* over $\breve{\mathfrak{M}}(\mathbf{G})$ to be

$$\mathcal{K}_{\breve{\mathfrak{M}}(\mathbf{G})} := \breve{F}^{\bullet}\mathcal{K}^{n}_{\mathfrak{M}(\mathbf{G})} + \mathcal{K}^{\circ}_{\breve{\mathfrak{M}}(\mathbf{G})}.$$

Then [3, Proposition 3.3] further implies that $\breve{F}$ is naturally an idealized log étale morphism of idealized log stacks

$$\breve{F}\colon (\breve{\mathfrak{M}}(\mathbf{G}), \mathcal{K}_{\breve{\mathfrak{M}}(\mathbf{G})}) \to (\mathfrak{M}, \mathcal{K}^{n}_{\mathfrak{M}(\mathbf{G})}). \tag{B.12}$$

Geometrically, this means that the obstruction to deforming punctured curves while keeping $\mathbf{G}$-marking is encoded in the puncturing log ideal $\mathcal{K}^{\circ}_{\breve{\mathfrak{M}}(\mathbf{G})}$.

B.1.11. *Tropical punctured curves.* Recall that for any geometric log point x, its tropicalization is the cone

$$\sigma_x := \mathrm{Hom}(\overline{\mathcal{M}}_x, \mathbb{R}_{\geq 0}).$$

Now consider a punctured curve $C^{\circ} \to S$ over a geometric log point S with $\overline{\mathcal{M}}_S = \sigma^{\vee}$. For each strict geometric point $z \to C^{\circ}$, we take the associated cone $\sigma_z := \mathrm{Hom}(\overline{\mathcal{M}}_{C^{\circ},z}, \mathbb{R}_{\geq 0})$. These cones are locally constant along logarithmic strata of C°, and glue to a cone complex $\Sigma(C^{\circ})$ via face morphisms, called the *tropicalization* of C°, see [3, Appendix C]. Furthermore, the functoriality of tropicalization yields a morphism of cone complexes

$$\Sigma(C^{\circ}) \to \Sigma(S) = \sigma \tag{B.13}$$

which we will describe explicitly in Construction B.4 below.

Definition B.3. A (possibly disconnected) *tropical punctured curve* (or simply *tropical curve*) over a cone $\sigma \in \mathbf{Cones}$ consists of a decorated graph $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$, a subset $\mathbf{L}^{\circ}(G) \subset \mathbf{L}(G)$, and a map

$$\ell\colon \mathbf{E}(G) \cup \mathbf{L}^{\circ}(G) \to \mathrm{Map}(\sigma, \mathbb{R}_{\geq 0}),$$

where $\ell(E) \in \sigma^{\vee}_{\mathbb{Z}} \setminus \{0\}$ for each edge $E \in \mathbf{E}(G)$, and $\ell(L)\colon \sigma \to \mathbb{R}_{\geq 0}$ is non-zero, concave, piece-wise linear, continuous with rational slopes for each leg $L \in \mathbf{L}^{\circ}(G)$. Legs in $\mathbf{L}^{\circ}(G)$ are referred to as *punctured legs*, while legs in $\mathbf{L}(G) \setminus \mathbf{L}^{\circ}(G)$ are referred to as *log legs*. The above family is called a *tropical log curve* if $\mathbf{L}^{\circ}(G) = \emptyset$.

Noting that the subset $\mathbf{L}^{\circ}(G) \subset \mathbf{L}(G)$ is determined by ℓ, we will use the notation $(\mathbf{G}, \ell)$ to denote a tropical curve over a cone σ.

Construction B.4. We recall from [3, §2.2.1] how to construct a morphism of cone complexes

$$\pi\colon \Gamma(G, \ell) \to \sigma \tag{B.14}$$

from the data (G, ℓ) of a tropical curve $(\mathbf{G}, \ell)$ as in Definition B.3. The fibers of this morphism are tropical curves in a more traditional sense.

To each $V \in \mathbf{V}(G)$, assign a cone $\sigma_V \cong \sigma$ with $\pi|_{\sigma}\colon \sigma_V \to \sigma$ the identity. To each $E \in \mathbf{E}(G)$, assign the cone

$$\sigma_E = \{(s, \lambda) \in \sigma \times \mathbb{R}_{\geq 0} \mid \lambda \leq \ell(E)(s)\} \tag{B.15}$$

with $\pi|_{\sigma_E}\colon \sigma_E \to \sigma$ the obvious projection. If V and V' are the two vertices attached to E (in some arbitrarily chosen order), define two face morphisms $\sigma_V \to \sigma_E$ and $\sigma_{V'} \to \sigma_E$ via

$$s \mapsto (s, 0) \quad \text{and} \quad s \mapsto (s, \ell(E)(s)),$$

respectively. These two face maps identify σ_V and $\sigma_{V'}$ with two facets of σ_E.

For each leg $L \in \mathbf{L}(G) \setminus \mathbf{L}^{\circ}(G)$ attached to a vertex V, assign the cone $\sigma_L = \sigma \times \mathbb{R}_{\geq 0}$ with the obvious projection $\pi|_{\sigma_L}\colon \sigma_L \to \sigma$, and the face morphism $\sigma_V \to \sigma_L$ identifying σ_V with the facet $\sigma \times \{0\}$.

For each punctured leg $L \in \mathbf{L}^\circ(G)$ attached to a vertex V, define

$$\sigma_L = \{(s,\lambda) \in \sigma \times \mathbb{R}_{\geq 0} \mid \lambda \leq \ell(L)(s)\}. \tag{B.16}$$

By [3, Lemma 2.21], σ_L is a cone. We further associate a face morphism $\sigma_V \to \sigma_L$ identifying σ_V with the facet $\sigma \times \{0\} \subset \sigma_L$, and define $\pi|_{\sigma_L} : \sigma_L \to \sigma$ the projection to the first factor. Punctured legs are precisely the legs with bounded length.

The cone complex $\Gamma(G,\ell)$ is defined as the colimit gluing cones $\{\sigma_V, \sigma_E, \sigma_L\}$ via the above face morphisms. Furthermore, the morphisms $\{\pi|_{\sigma_V}, \pi|_{\sigma_E}, \pi|_{\sigma_L}\}$ defined above glue to the morphism of cone complexes (B.14).

Note that the cone complex $\Gamma(G,\ell)$ and the morphism π do not depend on the data $\mathbf{g}, \mathbf{deg}$ and $\mathbf{m}$.

Construction B.5. Consider a punctured curve $C^\circ \to S$ over a geometric log point S with $\overline{\mathcal{M}}_S = \sigma_{\mathbb{Z}}^\vee$. Its *associated tropical curve* $(\mathbf{G}, \ell)$ over σ as in Definition B.3 is constructed as follows.

Let $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ be the decorated graph of the underlying pre-stable curve $\underline{\mathcal{C}} \to \underline{S}$. For each $E \in \mathbf{E}(G)$ its length $\ell(E) \in \sigma_{\mathbb{Z}}^\vee = \overline{\mathcal{M}}_S$ is the smoothing parameter of the corresponding node as in §B.1.3. Finally, for a punctured leg $L \in \mathbf{L}^\circ(G)$ corresponding to a marking $p \subset C^\circ$, by [3, Lemma 2.21] there is a unique non-zero, concave, piecewise linear function

$$\ell(p) \colon \sigma \to \mathbb{R}_{\geq 0}$$

with rational slopes such that

$$(\overline{\mathcal{M}}_{C^\circ}|_p)_{\mathbb{R}}^\vee = \{(s,\lambda) \in \sigma \times \mathbb{R}_{\geq 0} \mid 0 \leq \lambda \leq \ell(p)\}.$$

Then we set $\ell(L) = \ell(p)$.

Given a punctured curve $C^\circ \to S$ over a geometric log point S, we may construct the associated tropical curve $(\mathbf{G}, \ell)$. Then, by [3, §2.2.1], the morphism of cone complexes (B.13) is the morphism (B.14) from Construction B.4 applied to the tropical curve $(\mathbf{G}, \ell)$.

B.2. **Punctured maps and their tropicalizations.**

Punctured maps extend the notion of log maps by allowing possibly negative contact orders along markings. They are introduced in [3] to study boundaries of moduli of log maps, and are a key for us to study boundaries of the moduli of log R-maps.

B.2.1. *Punctured maps and the universal target $\mathcal{A}$.* A *punctured map* to a log stack X over an fs log scheme S is the data

$$(C^\circ \to S, f \colon C^\circ \to X)$$

where $C^\circ \to S$ is a punctured curve, and f is a morphism of log stacks. *Pull-backs* of punctured maps along morphisms of log schemes are defined via pull-backs of punctured curves.

For a punctured map $f \colon C^\circ \to X$ with DF1 target X as in §A.2.1, the composition $\mathfrak{f} \colon C^\circ \to X \to \mathcal{A}$, is called the *associated punctured map to $\mathcal{A}$*. As the morphism $X \to \mathcal{A}$ is strict, properties of f on the level of log structures is entirely captured by $\mathfrak{f}$. Thus for simplicity, we will focus on the universal target $\mathcal{A}$. Contact orders, degeneracies, the tropicalization and basicness of a punctured map $f \colon C^\circ \to X$ with X a DF1 target are defined and constructed in the same way as the corresponding notions for its associated punctured map to $\mathcal{A}$.

B.2.2. *Contact orders along markings.* Consider a punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over a log point S. Let $p_i \subset C^\circ$ be its i-th marking. By (B.5), along p_i we have

$$\overline{\mathcal{M}}_C|_{p_i} \cong \overline{\mathcal{M}}_S \oplus \mathbb{N} \subset \overline{\mathcal{M}}_{C^\circ}|_{p_i} \subset \overline{\mathcal{M}}_S \oplus \mathbb{Z}.$$

Consider the composition

$$u_{p_i}\colon \mathbb{N} = \Gamma(\mathcal{A}, \overline{\mathcal{M}}_{\mathcal{A}}) \xrightarrow{\bar{\mathfrak{f}}^\flat} \overline{\mathcal{M}}_{C^\circ}|_{p_i} \subset \overline{\mathcal{M}}_S \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$

where the last arrow is the projection. Note that the map u_{p_i} and the integer $\mathbf{c}(p_i) := u_{p_i}(1) \in \mathbb{Z}$ determine each other, and are both referred to as the *contact order* at p_i.

We say that $\mathfrak{f}$ is *pre-stable* along p_i if $\overline{\mathcal{M}}_{C^\circ}|_{p_i}$ is the fine (not necessarily saturated) monoid generated by $\overline{\mathcal{M}}_C|_{p_i}$ and $\bar{\mathfrak{f}}^\flat(1)$, see [3, Definition 2.6]. We call $\mathfrak{f}$ *pre-stable* if it is pre-stable along all markings.

Note that when $\mathbf{c}(p_i) \geq 0$, the pre-stability forces $\overline{\mathcal{M}}_{C^\circ}|_{p_i} = \overline{\mathcal{M}}_C|_{p_i}$. Consequently, the two log structures $\mathcal{M}_C$ and $\mathcal{M}_{C^\circ}$ only differ at markings with strictly negative contact orders. Thus we call p_i a *punctured marking* if $\mathbf{c}(p_i) < 0$, or a *log marking* if $\mathbf{c}(p_i) \geq 0$.

A punctured map to $\mathcal{A}$ over an fs log scheme S is said to be *pre-stable* if each geometric fiber is pre-stable. As shown in [3, Prop. 2.16] pre-stability is an open condition and is stable under pull-backs along morphisms of fs log schemes. In this paper, all punctured maps are assumed to be pre-stable unless otherwise specified.

B.2.3. *Contact orders along nodes.* Consider a punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over a log point S. Let $p \subset C^\circ$ be a node. By §B.1.3, along p we have

$$u_p\colon \mathbb{N} = \Gamma(\mathcal{A}, \overline{\mathcal{M}}_{\mathcal{A}}) \xrightarrow{\bar{\mathfrak{f}}^\flat} \overline{\mathcal{M}}_{C^\circ}|_p \cong \overline{\mathcal{M}}_S \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \overline{\mathcal{M}}_{C/S}|_p \cong \mathbb{Z}$$

where $\overline{\mathcal{M}}_{C/S} = \overline{\mathcal{M}}_C/\overline{\mathcal{M}}_S$ is the relative characteristic.

Note that $\overline{\mathcal{M}}_{C/S}|_p$ is generated by the images $\overline{(1,0)}, \overline{(0,1)}$ of $(1,0), (0,1) \in \overline{\mathcal{M}}_S \oplus_{\mathbb{N}} \mathbb{N}^2$ respectively, subject to the relation $\overline{(1,0)} + \overline{(0,1)} = 0$ by (B.3). Thus the isomorphism $\overline{\mathcal{M}}_{C/S}|_p \cong \mathbb{Z}$ depends on a choice of $\overline{(1,0)} \mapsto 1$ or $\overline{(1,0)} \mapsto -1$. In case $u_p(1) \neq 0$, we make the canonical choice $\overline{\mathcal{M}}_{C/S}|_p \cong \mathbb{Z}$ such that $u_p(1) > 0$. In any case the map u_p and the non-negative integer $u_p(1)$ determine each other, and will be referred to the *contact order* along the node p. This definition is compatible with the previous papers [16, 18, 20].

Let G be the dual graph of the underlying curve $\underline{C}$, and $E = \{h, \hat{h}\}$ be the edge corresponding to the node p. The positive integer $u_p(1)$ is also referred to as the *contact order* of E, denoted by $\mathbf{c}(E)$.

We may also define contact orders corresponding to the half-edges. First, if $u_p(1) = 0$ we define the contact orders of h and $\hat{h}$ to be $\mathbf{c}(h) = \mathbf{c}(\hat{h}) = 0$.

Now assume $u_p(1) > 0$. Without loss of generality, we may assume that the canonical choice $\overline{\mathcal{M}}_{C/S}|_p \cong \mathbb{Z}$ above identifies $\overline{(1,0)}$ with $1 \in \mathbb{Z}$. Note that $\overline{(1,0)}, \overline{(0,1)}$ come from the local coordinates of the two irreducible components intersecting at p, hence are naturally labeled by the two half-edges h and $\hat{h}$ such that the two irreducible components are labeled by $\nu_G(h)$ and $\nu_G(\hat{h})$, respectively. Then we define $\mathbf{c}(h) = u_p(1)$, and $\mathbf{c}(\hat{h}) = -u_p(1)$.

In particular, for every half-edge h belonging to an edge E, we have $\mathbf{c}(h) = -\mathbf{c}(\iota_G(h))$.

B.2.4. *Tropical maps to* $\mathbb{R}_{\geq 0}$. Abstractly, a *tropical punctured map* (or simply *tropical map*) to $\mathbb{R}_{\geq 0}$ over $\sigma \in \mathbf{Cones}$ is the data

$$(\mathbf{G}, \ell, f^{\mathrm{trop}}\colon \Gamma(G, \ell) \to \mathbb{R}_{\geq 0}) \tag{B.17}$$

where $(\mathbf{G}, \ell)$ is a (possibly disconnected) tropical curve as in Definition B.3, and f^{trop} is a morphism of cone complexes. We may also refer to $f^{\mathrm{trop}}\colon \Gamma(G, \ell) \to \mathbb{R}_{\geq 0}$ as a tropical map when there is no danger of confusion.

Recall that $\Sigma(\mathcal{A}) = \mathrm{Hom}(\mathbb{N}, \mathbb{R}_{\geq 0}) \cong \mathbb{R}_{\geq 0}$. Consider a punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over a log point S with $\sigma = \overline{\mathcal{M}}_S^\vee$. Tropicalizing $\mathfrak{f}$, we obtain

$$\Sigma(\mathfrak{f})\colon \Sigma(C^\circ) \to \Sigma(\mathcal{A}) = \mathbb{R}_{\geq 0}. \tag{B.18}$$

Thus the *associated tropical map* to $\mathfrak{f}$ is defined to be the data (B.17) where $(\mathbf{G}, \ell)$ is the tropical curve associated to $C^\circ \to S$, and $f^{\mathrm{trop}} = \Sigma(\mathfrak{f})$.

B.2.5. *Tropical types.* A *type* is a collection of the data

$$(\mathbf{G}, \boldsymbol{\sigma}, \mathbf{c}) \tag{B.19}$$

consisting of a decorated graph $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$, together with

(i) an *image cone map* $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}(x)$ is a face of $\mathbb{R}_{\geq 0}$ for any $x \in \mathbf{V}(G) \cup \mathbf{H}(G)$ satisfying $\boldsymbol{\sigma}(x) = \boldsymbol{\sigma} \circ \iota_G(x)$, and

(ii) a *contact order* map $\mathbf{c}\colon \mathbf{H}(G) \to \mathbb{Z}$ such that $\mathbf{c}(h) = -\mathbf{c}(\hat{h})$ for each edge $E = \{h, \hat{h}\}$.

Remark B.6. For an edge $E = \{h, \hat{h}\}$ note that $\boldsymbol{\sigma}(h) = \boldsymbol{\sigma}(\hat{h})$. Hence we may define $\boldsymbol{\sigma}(E) := \boldsymbol{\sigma}(h)$. In particular, the above definition is compatible with [3, §2.2.1].

For a tropical map (B.17) over σ, its tropical type (B.19) is defined as follows:

1. $\mathbf{G}$ is the decorated graph of the domain tropical curve as in §B.1.11.
2. For any $x \in \mathbf{V}(G) \cup \mathbf{L}(G)$, $\boldsymbol{\sigma}(x)$ is the minimal face of $\mathbb{R}_{\geq 0}$ containing $f^{\mathrm{trop}}(\sigma_x)$.
3. For any edge $E = \{h, \hat{h}\}$, $\boldsymbol{\sigma}(h) = \boldsymbol{\sigma}(\hat{h})$ is the minimal face of $\mathbb{R}_{\geq 0}$ containing $f^{\mathrm{trop}}(\sigma_E)$.
4. For each leg h, its *contact order* $\mathbf{c}(h) \in \mathbb{Z}$ is defined as the image of the tangent vector $(0, 1) \in \sigma \times \mathbb{R}_{\geq 0}$ by f^{trop}.
5. For each half-edge h forming an edge $\{h, \hat{h}\}$ with $\nu_G(h) = V$, the contact order $\mathbf{c}(h)$ is defined as the image of the tangent vector $(0, 1) \in \sigma \times \mathbb{R}_{\geq 0}$ by f^{trop}.

A type (B.19) is called *realizable* if it is a type of a tropical map to $\mathbb{R}_{\geq 0}$. For a punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over a log point S, we define its *tropical type* to be the type of its associated tropical map. It is straightforward to check that the contact orders of half-edges is compatible with the definitions in §B.2.2 and §B.2.3.

Remark B.7. Let h be a half-edge forming an edge $E = \{h, \hat{h}\}$. If $\mathbf{c}(h) = 0$ then $\mathbf{c}(\hat{h}) = 0$, in which case we define $\mathbf{c}(E) = 0$ as well. If $\mathbf{c}(h) \neq 0$, then by §B.2.3 we have $\mathbf{c}(E)$ is equal to the unique positive integer in $\{\mathbf{c}(h), \mathbf{c}(\hat{h})\}$. Thus, we observe that the contact orders $\mathbf{c}(h), \mathbf{c}(\hat{h})$ and $\mathbf{c}(E)$ determine each other.

Notation B.8. Since $\mathbb{R}_{\geq 0}$ has only two faces, an element $x \in \mathbf{V}(G) \cup \mathbf{H}(G)$ is called *non-degenerate* if $\boldsymbol{\sigma}(x) = 0$, and is called *degenerate* otherwise. We will frequently use these terminologies without specifying $\boldsymbol{\sigma}$.

B.2.6. *Degeneracies and their partial ordering.* Consider a tropical map $f^{\mathrm{trop}}\colon \Gamma(G, \ell) \to \mathbb{R}_{\geq 0}$ over σ. For each $V \in \mathbf{V}(G)$, the restriction $e_V := f^{\mathrm{trop}}|_{\sigma_V}\colon \sigma_V \to \mathbb{R}_{\geq 0}$ is called the *degeneracy* of the vertex V. Since $\sigma_V \cong \sigma$, the morphism e_V is naturally an element of the monoid $\sigma_{\mathbb{Z}}^\vee$. The finite set of degeneracies $\{e_V\}_{V \in \mathbf{V}(G)}$ as a subset of $\sigma_{\mathbb{Z}}^\vee$ is partially ordered with respect to $\preccurlyeq_{\sigma^\vee}$. This defines a partial order on $\mathbf{V}(G)$ such that $V_1 \preccurlyeq V_2$ if $e_{V_1} \preccurlyeq_{\sigma^\vee} e_{V_2}$. Note that if E is an edge joining V_1 to V_2 such that $V_1 \preccurlyeq V_2$ and not contracted by f^{trop}, then the contact order $\mathbf{c}(E) > 0$ by §B.2.3.

Consider a punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over a log point S with $\overline{\mathcal{M}}_S = \sigma_{\mathbb{Z}}^\vee$. Let $\Sigma(\mathfrak{f})\colon \Gamma(G, \ell) \to \mathbb{R}_{\geq 0}$ be the corresponding tropicalization over σ. The degeneracy e_V of a vertex $V \in \mathbf{V}(G)$ can be also described directly as follows.

Let $Z_V \subset C^\circ$ be the irreducible component corresponding to V, and $x \in Z_V$ be a smooth unmarked point. Then e_V is the image of 1 via the above composition

$$\mathbb{N} \longrightarrow \mathfrak{f}^*\overline{\mathcal{M}}_{\mathcal{A}}|_x \xrightarrow{\mathfrak{f}^\flat} \overline{\mathcal{M}}_{C^\circ}|_x \cong \overline{\mathcal{M}}_S.$$

We recall that degeneracies and their partial orderings are stable under generizations:

Lemma B.9. *Consider a punctured map* $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ *over a connected log scheme* S. *For each geometric point* $t \in S$, *denote by* $\Sigma(\mathfrak{f})_t\colon \Gamma(G_t, \ell_t) \to \mathcal{A}$ *over* $\sigma_t = \overline{\mathcal{M}}_S|_t^\vee$ *the tropicalization of the fiber* $\mathfrak{f}|_t$. *Suppose there is a chart* $\phi : \overline{\mathcal{M}}_{S,t} \to \mathcal{M}_S$ *for a strict log point* $s \to S$. *Then*

1. *For any pair* $e_1, e_2 \in \overline{\mathcal{M}}_S|_s$ *satisfying* $e_1 \preccurlyeq e_2$, *we have* $\phi(e_1)|_t \preccurlyeq \phi(e_2)|_t$ *for any* $t \in S$.
2. *If* $e \in \overline{\mathcal{M}}_S|_s$ *is a degeneracy, then there is a strict étale neighborhood* $U \to S$ *of* s *such that* $\phi(e)|_t$ *is a degeneracy for any* $t \in U$.

Proof. This is proved in [20, Lemma 3.4, 3.6] for log maps. As the proof relies only on the smooth non-marked locus of $C^\circ \to S$, it applies identically to the punctured maps case. ☐

B.2.7. *Pre-stability and the puncturing log-ideal.* A tropical map $f^{\mathrm{trop}}\colon \Gamma(G,\ell) \to \mathbb{R}_{\geq 0}$ over σ is called *pre-stable* if, in the notations of (B.16), any $L \in \mathbf{L}^\circ(G)$ satisfies $\mathbf{c}(L) < 0$ and $f^{\mathrm{trop}}((s, \ell(L)(s))) = 0$ for any $(s, \ell(L)(s)) \in \sigma_L$. This means that if $L \in \mathbf{L}^\circ(G)$ then L is degenerate, and the image $f^{\mathrm{trop}}(\sigma_L)$ as a finite ray of non-zero length extends to the boundary $0 \in \mathbb{R}_{\geq 0}$. In this paper, all tropical maps are assumed to be pre-stable unless otherwise specified. Note that the prestability implies that the subset $\mathbf{L}^\circ(G) \subset \mathbf{L}(G)$ is the collection of legs with negative contact orders, hence is specified by $\mathbf{c}$. For any $L \notin \mathbf{L}^\circ(G)$, we necessarily have $\mathbf{c}(L) \geq 0$ as log legs are unbounded.

Consider a punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over a log point S with $\overline{\mathcal{M}}_S = \sigma_{\mathbb{Z}}^\vee$, and let $\Sigma(\mathfrak{f})\colon \Gamma(G,\ell) \to \mathbb{R}_{\geq 0}$ over σ be the associated tropical map of $\mathfrak{f}$. By [3, Prop. 2.23], the pre-stability of $\mathfrak{f}$ translates to the pre-stability of the associated tropical map. Assuming prestability, the puncturing log-ideal $\mathcal{K}_S^\circ$ has a nice description:

Define the *puncturing monoid ideal* of $\overline{\mathcal{M}}_S$:

$$\overline{\mathcal{K}}_S^\circ := \langle e_V \mid V \in \mathbf{V}(G) \text{ and } V \text{ has a punctured leg } L \in \mathbf{L}^\circ(G) \rangle.$$

By the description of the puncturing log-ideal in §B.1.5, the pre-stability of §B.2.2 implies that the puncturing log ideal is given by $\mathcal{K}_S^\circ = \mathcal{M}_S \times_{\overline{\mathcal{M}}_S} \overline{\mathcal{K}}_S^\circ$. Furthermore, the condition $\ell(L) \neq 0$ for each $L \in \mathbf{L}^\circ(G)$ forces $e_V \neq 0$ for degeneracies $e_V \in \overline{\mathcal{K}}_S^\circ$. We refer to [3, §2.5] for a more comprehensive discussion on this.

B.2.8. *The balancing condition.* The *balancing condition* is an important constraint that controls the combinatorial structure of punctured maps.

Consider a tropical map $(G, \mathbf{g}, \mathbf{deg}, \mathbf{m}, \ell, f^{\mathrm{trop}})$ over a cone σ. The function $\mathbf{deg}$ can be extended to vertices via

$$\mathbf{deg}(V) = \sum_{h \in \mathbf{H}(V)} \mathbf{c}(h)\mathbf{deg}(h) \tag{B.20}$$

for any $V \in \mathbf{V}(G)$, defining a function $\mathbf{deg}\colon \mathbf{V}(G) \sqcup \mathbf{H}(G) \longrightarrow \mathbb{Q}$.

Let $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ be a punctured map over a log point S, and $(G, \mathbf{g}, \mathbf{deg}, \mathbf{m}, \ell, \Sigma(\mathfrak{f}))$ be the corresponding tropical map over $\sigma := \overline{\mathcal{M}}_S^\vee$. For any vertex $V \in \mathbf{V}(G)$, let $Z_V \subset C^\circ$ be the corresponding irreducible component. As shown in [3, §2.2.3], the restriction $\mathfrak{f}|_{Z_V}$ induces an isomorphism of line bundles

$$\mathfrak{f}^*\mathcal{O}(\infty_{\mathcal{A}})|_{Z_V} \cong \mathcal{O}_{Z_V}\Big(\sum_{h \in \mathbf{H}(V)} \mathbf{c}(h) p_h\Big). \tag{B.21}$$

Taking the degree on both sides, we obtain the *balancing condition*

$$\deg \mathfrak{f}^*\mathcal{O}(\infty_{\mathcal{A}})|_{Z_V} = \mathbf{deg}(V). \tag{B.22}$$

B.3. **Stacks of punctured maps to $\mathcal{A}$.** The stacks of punctured maps to $\mathcal{A}$ interpolates between the moduli of R-maps and their tropical counterparts, and play important roles in the tropical study of this paper.

B.3.1. *Basicness.* The key to the stack of punctured maps is the notion of *basic punctured maps*, which translates to the notion of universality on the tropical side.

Definition B.10. A tropical map $(\mathbf{G}, \ell, f^{\mathrm{trop}})$ over $\sigma \in \mathbf{Cones}$ is *basic* if it is universal among all tropical maps with the same type. More precisely, if another tropical map $(\mathbf{G}', \ell', f^{\mathrm{trop},\prime})$ over a cone σ' is of the same type, then there is a unique morphism of cones $\sigma' \to \sigma$ realizing $(\mathbf{G}', \ell', f^{\mathrm{trop},\prime})$ as the pull-back of $(\mathbf{G}, \ell, f^{\mathrm{trop}})$.

Construction B.11. For a tropical map $(\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}), \ell, f^{\mathrm{trop}})$ over $\sigma \in \mathbf{Cones}$, its associated basic tropical map is constructed in [3, §2.3]. We summarize it in our situation as follows.

The associated *basic cone* σ_{bas} is defined by its lattice points

$$\sigma_{bas,\mathbb{Z}} := \{((p_V)_V, (e_E)_E) \in \prod_{V \in \mathbf{V}(G)} \boldsymbol{\sigma}(V)_{\mathbb{Z}} \times \prod_{E \in \mathbf{E}(G)} \mathbb{N} \mid p_V - p_{V'} = \mathbf{c}(E') \cdot e_{E'}\} \tag{B.23}$$

where the equation holds for all triples (E', V', V) where E' is an edge with two ends V' and V. The associated *basic monoid* is defined to be its dual $\sigma^\vee_{bas,\mathbb{Z}}$. The above construction leads to a natural morphism

$$\sigma \to \sigma_{bas}, \qquad m \mapsto ((e_V(m))_V, (\ell(E)(m))_E), \tag{B.24}$$

hence a dual on the monoid level

$$\sigma^\vee_{bas,\mathbb{Z}} \to \sigma^\vee_{\mathbb{Z}}. \tag{B.25}$$

By [3, Proposition 2.32], there is a well-defined *basic tropical map*

$$f^{\mathrm{trop}}_{bas} \colon \Gamma(G, \ell_{bas}) \to \mathbb{R}_{\geq 0}$$

over σ_{bas} with the domain tropical curve $(\mathbf{G}, \ell_{bas})$, whose pullback along the natural morphism (B.24) is f^{trop}.

We briefly review the construction of f^{trop}_{bas} in our situation. We first define the edge lengths

$$\ell_{bas} \colon \mathbf{E}(G) \to \mathrm{Hom}(\sigma_{bas}, \mathbb{N}) \setminus \{0\}, \qquad \ell_{bas}(E)\,((p_V)_V, (e_{E'})_{E'}) = e_E.$$

For any $V \in \mathbf{V}(G)$, its degeneracy with respect to f^{trop}_{bas} is

$$e_V := f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,V}} \colon \sigma_{bas,V} = \sigma_{bas} \to \mathbb{R}_{\geq 0}, \qquad ((p_{V'})_{V'}, (e_E)_E) \mapsto p_V.$$

For any $E \in \mathbf{E}(G)$ attached to a vertex V, the restriction $f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,E}}$ is constructed from e_V, $\mathbf{c}(E)$, and $\ell(E)$. First, recall that $\sigma_{bas,E}$ is defined from its facet $\sigma_{bas,V}$ and the edge length $\ell(E)$ via (B.15). Furthermore, the restriction of $f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,E}}$ to the facet $\sigma_{bas,V} \subset \sigma_{bas,E}$ must be e_V. We then extend $f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,E}}$ linearly to the whole cone $\sigma_{bas,E}$ with slope $\mathbf{c}(E)$.

Similarly for a log leg $L \in \mathbf{L}(G)$ attached to a vertex V, since L is unbounded, we construct the restriction $f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,L}}$ by extending $e_V = f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,V}}$ linearly over $\sigma_{bas,L}$ with slope $\mathbf{c}(L)$.

For a punctured leg $L \in \mathbf{L}^\circ(G)$ attached to a vertex V, the cone σ_L and the restriction $f^{\mathrm{trop}}_{bas}|_{\sigma_L}$ are uniquely determined by the degeneracy $f^{\mathrm{trop}}_{bas}|_{\sigma_{bas,V}}$ of V and the contact order $\mathbf{c}(L)$ thanks to the prestability in §B.2.7. In particular, this defines the length $\ell_{bas}(L)$ for the punctured leg L.

Remark B.12. Note that the basic cone (B.23) and the natural morphism (B.24) depend only on the data $(G, \boldsymbol{\sigma}, \mathbf{c})$, and are independent of $\mathbf{g}$ and $\mathbf{deg}$.

Definition B.13. A punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S is *basic* if for every geometric fiber, the associated tropical map is basic.

The nice properties enjoyed by basic punctured maps are summarized in Propositions B.14, B.15 and B.16 below.

Proposition B.14. *For any pre-stable punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S, the locus with basic fibers forms an open subset of S.*

Proof. Note that basicness is a property of the characteristic sheaf $\overline{\mathcal{M}}_S$ over the base S. The proposition then follows by the same proof as in the case of log maps [16, Prop. 3.5.2], see also [3, Prop. 2.34]. ☐

Proposition B.15. *Any pre-stable punctured map to $\mathcal{A}$ arises as the pull-back from a basic pre-stable punctured map to $\mathcal{A}$ with the same underlying pre-stable map. Both the basic pre-stable punctured map and the morphism are unique up to a unique isomorphism.*

Proof. When there are no punctures, this is [20, Prop. 2.8]. In the presence of punctures but no stacky structures on the domain curve, this is a special case of [3, Prop 2.35]. We here observe that the same proof applies to the case of twisted curves:

First, to produce the basic family, we do not modify the underlying structure, but we modify the characteristic sheaves of monoids on both the domain curves and their bases. The proof of [3, Prop 2.35] constructs basic families by modifying the base characteristic monoids using étale local constraints from the characteristic monoids of domain curves and targets. As the log structures of the domain curves admit charts étale locally, the corresponding characteristic sheaves are étale sheaves. Hence the proof of [3, Prop 2.35] applies to our situation of orbifold domain curves identically. ☐

Lemma B.16. *An automorphism of a basic punctured map fixing the underlying curve, is trivial.*

Proof. This is identical to the case of log maps [16, Lem. 3.8.3] and [24, Prop. 1.25], see also [3, Prop. 2.37]. ☐

B.3.2. *Contractions.* Next, we recall *contractions of types* following [3, §2.2.2]. Geometrically, they correspond to smoothing of punctured maps.

Consider two types (B.19) of tropical maps

$$\tau = (\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}), \boldsymbol{\sigma}, \mathbf{c}) \quad \text{and} \quad \tau' = (\mathbf{G}' = (G', \mathbf{g}', \mathbf{deg}', \mathbf{m}'), \boldsymbol{\sigma}', \mathbf{c}').$$

A *contraction* $\phi\colon \tau' \to \tau$ is a contraction of decorated graphs $\phi\colon \mathbf{G}' \to \mathbf{G}$ as in §B.1.6, satisfying the additional compatibilities

(i) The cone $\boldsymbol{\sigma}(\phi(x))$ is a face of $\boldsymbol{\sigma}'(x)$ for any $x \in \mathbf{V}(G') \cup \mathbf{H}(G')$.
(ii) $\mathbf{c}(\mathbf{L}(\phi)(L)) = \mathbf{c}'(L)$ for any $L \in \mathbf{L}(G')$, and $\mathbf{c}(E) = \mathbf{c}'(\mathbf{E}(\phi)(E))$ for any $E \in \mathbf{E}(G)$.

Note that (i), is equivalent to the condition that for any $x \in \mathbf{V}(G') \cup \mathbf{H}(G')$, $\phi(x)$ is degenerate only if x is degenerate.

Let $\phi\colon \tau' \to \tau$ be a contraction of types, and let σ and σ' be the basic cones of τ and τ' respectively. By [3, Remark 2.46] the contraction ϕ induces a natural face morphism of basic cones

$$\chi_\phi^\vee\colon \sigma \to \sigma' \tag{B.26}$$

such that a point $(p_V, e_E)_{V,E} \in \sigma'$ is contained in σ iff

(1) $p_V \in \boldsymbol{\sigma}(\mathbf{V}(\phi)(V))$ for any $V \in \mathbf{V}(G')$,
(2) $e_E = 0$ for any $E \in \mathbf{E}_\phi$.

Taking duals, this leads to a morphism of basic monoids [3, (2.20)]

$$\chi_\phi\colon (\sigma'_{\mathbb{Z}})^\vee \to \sigma^\vee_{\mathbb{Z}} \tag{B.27}$$

satisfying

($1^\vee$) $\chi_\phi(e_V) = 0$ if $\boldsymbol{\sigma}(V(\phi)(V)) = \{0\} \subset \mathbb{R}_{\geq 0}$,
($2^\vee$) $\chi_\phi(\ell(E)) = 0$ if $E \in \mathbf{E}_\phi$.

The τ-*basic monoid ideal* is defined to be

$$\overline{\mathcal{K}}_\phi := \chi_\phi^{-1}(\sigma^\vee_{\mathbb{Z}} \setminus \{0\}) \subset (\sigma')^\vee_{\mathbb{Z}} \tag{B.28}$$

Definition B.17. Let τ be a type. A τ-*marking* of a basic tropical map of type τ' is a contraction $\phi\colon \tau' \to \tau$ of types.

Notation B.18. We may use the notation $\tau' \vdash \tau$ for a contraction $\phi\colon \tau' \to \tau$ when we want to emphasize the types without specifying ϕ.

B.3.3. *Punctured maps marked by types.* Fix a type $\tau = (\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}), \boldsymbol{\sigma}, \mathbf{c})$. A basic punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S is said to be *weakly marked* by τ if

1. The underlying of the domain curve $C^\circ \to S$ is marked by $\mathbf{G}$.
2. For any $x \in \mathbf{V}(G) \cup \mathbf{H}(G)$ let $Z_x \subset C^\circ$ be the corresponding substack. Then $f(Z_x)$ factors through $0_{\mathcal{A}} \subset \mathcal{A}$ if x is degenerate, see Notation B.8.
3. For any geometric point $s \in S$, let $\tau_s = (\mathbf{G}_s, \boldsymbol{\sigma}_s, \mathbf{c}_s)$ be the tropical type of the fiber $\mathfrak{f}|_s$. The contraction $\mathbf{G}_s \to \mathbf{G}$ given by the marking of the domain curves induces a contraction of types $\phi_{\tau_s,\tau}\colon \tau_s \to \tau$.

A weak marking of f by τ is said to be a *marking* by τ if the further condition hold:

4. For any geometric point $s \to S$ as above, the log-ideal $\mathcal{K}_{\phi_{\tau_s,\tau}} \subset \mathcal{M}_{S,s}$ defined as the pre-image of the basic monoid ideal $\overline{\mathcal{K}}_{\phi_{\tau_s,\tau}} \subset \overline{\mathcal{M}}_{S,s}$ maps to 0 under the structure morphism $\mathcal{M}_{S,s} \to \mathcal{O}_{S,s}$.

Remark B.19. The difference between weakly markings and markings by types are very subtle, and is explained carefully in [3, Remark 3.5]. The reason for introducing weakly markings is because they naturally appear in the gluing of punctured maps. In this paper, weakly markings will be only used in §5.5 but in a crucial way. In what follows, we will include discussions regarding weakly markings, markings and their relations. Readers who wish to skip the technical details may skip the notion of weakly markings for the moment, and return back to this section when needed.

As in [3, §3.5.1], a marking and a weak marking by τ lead to different idealized structures on the base S. Consider a strict geometric point $s \in S$, with $\Sigma(\mathfrak{f}_s)\colon \Gamma(G_s, \ell_s) \to \mathbb{R}_{\geq 0}$ the tropicalization of the fiber $\mathfrak{f}_s$. For any $x \in \mathbf{V}(G_s) \cup \mathbf{E}(G_s) \cup \mathbf{L}(G_s)$, denote by $Z_x \subset C^\circ$ the corresponding substack, and $z \in Z_x$ a geometric point. If x is a vertex, we require z to be a smooth unmarked point of $\underline{C}$. We have a sequence of morphisms of monoids

$$\begin{tikzcd}
\overline{\mathcal{M}}_S|_s \cong \overline{\mathcal{M}}_S|_z \arrow[r, "\bar{\pi}_s^\flat"] & \overline{\mathcal{M}}_C|_z & \boldsymbol{\sigma}(x)^\vee_{\mathbb{Z}} \arrow[l, "\bar{\mathfrak{f}}_s^\flat"'] \arrow[r, "\chi_x"] & \boldsymbol{\sigma}(\phi_{\tau_s,\tau}(x))^\vee_{\mathbb{Z}}
\end{tikzcd}$$

where χ_x is the dual of the face map in §B.3.2 (i). Define the set of *target stratum generators* with respect to x to be

$$(\bar{\pi}_s^\flat)^{-1}\left(\bar{\mathfrak{f}}_s^\flat\left(\boldsymbol{\sigma}(x)^\vee_{\mathbb{Z}} \setminus \chi_x^{-1}(0)\right)\right) \tag{B.29}$$

The *weak τ-marking monoid ideal* $\overline{\mathcal{K}}^w_{S,\tau}$ is a sheaf of ideals in $\overline{\mathcal{M}}_S$ with stalk at a strict geometric point $s \in S$ generated by

($1'$) the nodal monoid ideal by pulling-back (B.10);
($2'$) the target stratum generators as in (B.29) for any $x \in \mathbf{V}(G_s) \cup \mathbf{E}(G_s) \cup \mathbf{L}(G_s)$.

The *τ-marking monoid ideal* $\overline{\mathcal{K}}_{S,\tau}$ is a sheaf of ideals in $\overline{\mathcal{M}}_S$ with stalk at a strict geometric point $s \in S$ generated by $(1')$, $(2')$ above and

$(3')$ the basic monoid ideal $\overline{\mathcal{K}}_{\phi_{\tau_s,\tau}}$ as in §B.3.2.

The *weak τ-marking log-ideal* $\mathcal{K}^w_{S,\tau}$ and the *τ-marking log-ideal* $\mathcal{K}_{S,\tau}$ are defined to be the preimage of $\overline{\mathcal{K}}^w_{S,\tau}$ and $\overline{\mathcal{K}}_{S,\tau}$ via the quotient $\mathcal{M}_S \to \overline{\mathcal{M}}_S$, respectively. By [3, Remark 2.36, Lemma 2.47], since $\mathcal{M}_S$ is fs, the log-ideals $\mathcal{K}^w_{S,\tau}$ and $\mathcal{K}_{S,\tau}$ are both coherent.

The same calculation as in [3, Def. 3.20] shows that

$$\alpha(\mathcal{K}^w_{S,\tau}) = 0, \qquad (\text{resp. } \alpha(\mathcal{K}_{S,\tau}) = 0) \tag{B.30}$$

if $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S is a weakly τ-marked (resp. τ-marked) basic punctured map. In particular $(S, \mathcal{K}^w_{S,\tau})$ (resp. $(S, \mathcal{K}_{S,\tau})$) becomes an idealized log scheme.

Remark B.20. The vanishing in (B.30) corresponding to elements in $(2')$ enforces the factorizations of (2). In particular, if $\phi_{\tau_s,\tau}(x)$ is non-degenerate, then the set (B.29) is empty. In general, the idealized structures (B.30) are constraints imposed by the combinatorial data of types.

Further recall the puncturing log-ideal $\mathcal{K}^\circ_S$ of the family $C^\circ \to S$. The *canonical idealized structure* of a weak τ-marking (resp. τ-marking) of a basic punctured map $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S is the log ideal generated by

$$\mathcal{K}^w_{S,\tau} + \mathcal{K}^\circ_S, \qquad (\text{resp. } \mathcal{K}_{S,\tau} + \mathcal{K}^\circ_S). \tag{B.31}$$

Remark B.21. It is useful to observe that when $\mathbf{E}(G) = \emptyset$, the two log-ideals in (B.31) agrees, hence the notions τ-marking and weakly τ-marking coincide. Indeed, in this case one checks that the basic monoid ideal in $(3')$ above is generated by the target stratum generators in (B.29).

In case τ is realizable, condition $(3')$ governs both $(1')$ and $(2')$:

Lemma B.22. *If τ is realizable, then the sheaf of monoid ideals $\overline{\mathcal{K}}_{S,\tau} + \overline{\mathcal{K}}^\circ_S$ given by (B.31) is fiberwise generated by basic monoid ideals as in $(3')$ above.*

Proof. The case of non-stacky domain curves is established in [3, Prop. 3.24] by analyzing the generization of basic monoids. By Remark B.12, the same proof applies identically to stacky domain curves. ☐

B.3.4. *Stacks of punctured maps marked by types.* Let $\tau = (\mathbf{G}, \boldsymbol{\sigma}, \mathbf{c})$ be a type. Consider the categories fibered over fs log schemes

$$\mathfrak{M}(\mathcal{A}, \tau), \qquad \mathfrak{M}'(\mathcal{A}, \tau) \tag{B.32}$$

parameterizing punctured maps to $\mathcal{A}$ marked and weakly marked by τ respectively. The next few propositions summarize properties of $\mathfrak{M}(\mathcal{A}, \tau)$ and $\mathfrak{M}'(\mathcal{A}, \tau)$.

Proposition B.23. *(1) Both $\mathfrak{M}(\mathcal{A}, \tau)$ and $\mathfrak{M}'(\mathcal{A}, \tau)$ are represented by algebraic stacks locally of finite type with their basic fs log structures.*

(2) The tautological morphisms

$$\mathfrak{M}(\mathcal{A}, \tau) \to \mathfrak{M}(\underline{\mathcal{A}}), \qquad \mathfrak{M}'(\mathcal{A}, \tau) \to \mathfrak{M}(\underline{\mathcal{A}}),$$

to the stack $\mathfrak{M}(\underline{\mathcal{A}})$ of usual pre-stable maps to $\underline{\mathcal{A}}$ are representable.

Proof. Statement (1) follows from applying Step 1 of [3, Theorem 3.10] to the case of orbifold domain curves. The representability in (2) is a consequence of Lemma B.16. ☐

Remark B.24. Let $\tau = (\mathbf{G}, \boldsymbol{\sigma}, \mathbf{c})$ be a type with $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$, and let $G = \cup G_i$ be the decomposition into connected components. Then τ naturally induces a collection of types $\{\tau_i\}_i$ with τ_i obtained by restricting τ to G_i. Note that any punctured map $f\colon C^\circ \to \mathcal{A}$ over S marked (resp. weakly marked) by τ, is equivalent to a collection of punctured maps $\{f_i \colon C_i^\circ \to \mathcal{A}\}$ over S such that $C^\circ = \sqcup_i C_i^\circ$, and $f_i = f|_{C_i^\circ}$ is marked (resp. weakly marked) by τ_i. Consequently, we have isomorphisms of log stacks

$$\mathfrak{M}(\mathcal{A},\tau) \longrightarrow \prod_i \mathfrak{M}(\mathcal{A},\tau_i), \qquad \mathfrak{M}'(\mathcal{A},\tau) \longrightarrow \prod_i \mathfrak{M}'(\mathcal{A},\tau_i).$$

Remark B.25. A (not necessarily realizable) type $\tau = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m}, \boldsymbol{\sigma}, \mathbf{c})$ is called a *class* of tropical maps to $\mathcal{A}$ if $\mathbf{V}(G) = \{V_0\}$, $\mathbf{E}(G) = \emptyset$, and $\boldsymbol{\sigma}(x) = \{0\}$ for any $x \in G$. Note that the condition $\boldsymbol{\sigma}(x) = \{0\}$ for any $x \in G$ implies that the constraint §B.3.3 (2) is trivial.

For a class τ as above, we may write $\varsigma = (\mathbf{deg}, \mathbf{c})$ so that $\varsigma(G) = (\mathbf{deg}(L), \mathbf{c}(L))$ for each $L \in \mathbf{L}(G)$, and write $g := \mathbf{g}(V_0)$. Then the tropical class τ is equivalent to the data of (g, ς). Further observe that τ has the basic cone $\sigma = \{0\}$, hence the constraint §B.3.3 (4) is trivial and $\mathfrak{M}'(\mathcal{A},\tau) = \mathfrak{M}(\mathcal{A},\tau)$. We introduce

$$\mathfrak{M}_{g,\varsigma}(\mathcal{A}) := \mathfrak{M}(\mathcal{A},\tau), \tag{B.33}$$

which is the stack of punctured maps with connected genus g domain curves with orbifold structure and contact orders specified by ς.

Indeed, for any punctured map $f \colon C^\circ \to \mathcal{A}$ over S of tropical type τ' with genus g connected domain satisfying the constraints $\varsigma = (\mathbf{deg}(L), \mathbf{c}(L))$, there is a unique contraction $\tau' \to \tau$ by contracting all edges. Now observe that f satisfies §B.3.3 (1), (3), hence is obtained by a unique morphism $S \to \mathfrak{M}_{g,\varsigma}(\mathcal{A})$.

B.3.5. *Idealized structures on stacks of punctured maps.* Let $\mathbf{G} = (G, \mathbf{g}, \mathbf{deg}, \mathbf{m})$ and consider the idealized log stack $(\mathfrak{M}(\mathbf{G}), \mathcal{K}^n_{\mathfrak{M}(\mathbf{G})})$ of pre-stable curves marked by $\mathbf{G}$ as in §B.1.8. Consider the tautological morphisms of idealized log stacks

$$\begin{aligned} (\mathfrak{M}(\mathcal{A},\tau), \mathcal{K}_{\mathfrak{M}(\mathcal{A},\tau)}) &\to (\mathfrak{M}(\mathbf{G}), \mathcal{K}^n_{\mathfrak{M}(\mathbf{G})}), \\ (\mathfrak{M}'(\mathcal{A},\tau), \mathcal{K}_{\mathfrak{M}'(\mathbf{G})(\mathcal{A},\tau)}) &\to (\mathfrak{M}(\mathbf{G}), \mathcal{K}^n_{\mathfrak{M}(\mathbf{G})}) \end{aligned} \tag{B.34}$$

by removing the data of punctured maps, where the log-ideals $\mathcal{K}_{\mathfrak{M}(\mathcal{A},\tau)}$ and $\mathcal{K}_{\mathfrak{M}'(\mathcal{A},\tau)}$ are the log-ideals defined in (B.31) respectively.

Proposition B.26. *Both morphisms in* (B.34) *are idealized log étale.*

Proof. The proof is identical to [3, Thm. 3.25]. Indeed, a key is the observation of [7] that $\mathcal{A}$ is log étale over Spec $\mathbb{C}$. In particular, punctured maps to $\mathcal{A}$ even with orbifold domains, are unobstructed, except the obstructions to preserving the combinatorial structures of τ-marking or weak τ-marking, which are encoded precisely by the corresponding idealized structures. ☐

Proposition B.27. *The tautological morphism*

$$\mathfrak{M}(\mathcal{A},\tau) \to \mathfrak{M}'(\mathcal{A},\tau) \tag{B.35}$$

is a strict closed embedding defined by a nilpotent ideal. In particular, the two stacks have the same reduced log stack.

Proof. Note that marking and weak marking by τ differ by the monoid ideal in §B.3.3 (3′). This statement follows from [3, Prop. 3.33], which shows that the τ-marking log ideal is contained in the radical of the weak τ-marking log ideal. ☐

We are mostly interested in the case that τ is realizable. In this case, the idealized log étaleness of (B.34) allows us to compute the dimensions.

Proposition B.28. *Suppose that τ is realizable with connected underlying graph G. Then $\mathfrak{M}(\mathcal{A},\tau)$ is non-empty, reduced, and pure-dimensional of dimension*

$$\dim \mathfrak{M}(\mathcal{A},\tau) = 3\mathbf{g}(G) - 3 + |L(G)| - \dim \sigma \tag{B.36}$$

where σ is the basic cone associated to the realizable type τ.

As $\mathfrak{M}(\mathcal{A}, \tau)$ and $\mathfrak{M}'(\mathcal{A}, \tau)$ have the same reduced stack, $\mathfrak{M}'(\mathcal{A}, \tau)$ is non-empty and pure-dimensional of the same dimension (B.36) for a connected and realizable type τ.

Proof. This follows from the same analysis as in [3, Remark 3.27 and Prop. 3.29]. The key is to apply [2, Prop. 2.10] and observe that punctured maps to $\mathcal{A}$ are determined by the corresponding tropical maps. ☐

B.4. **Punctured maps to a log stack.** Let X be a separated log Deligne-Mumford stack with a strict morphism $X \to \mathcal{A}$. A *punctured map to* X over an fs base S is a morphism $f\colon C^\circ \to X$ where $C^\circ \to S$ is a punctured curve. The *associated punctured map to* $\mathcal{A}$ is $\mathfrak{f}\colon C^\circ \to \mathcal{A}$ over S obtained by composing f with $X \to \mathcal{A}$. We call f *stable* if the underlying pre-stable map $\underline{f}$ is stable in the usual sense. In particular, $\underline{f}$ is representable.

Fix a type τ as in (B.19), consider the categories of stable punctured maps to X

$$\mathscr{M}(X, \tau), \qquad \mathscr{M}'(X, \tau)$$

whose associated punctured maps to $\mathcal{A}$ are marked and weakly marked by τ respectively. We collect various properties of these two stacks as needed in this paper.

B.4.1. *Representability.*

Proposition B.29. *Both $\mathscr{M}(X, \tau)$ and $\mathscr{M}'(X, \tau)$ are represented by log Deligne-Mumford stacks locally of finite type.*

Proof. Lemma B.16 implies that stable punctured maps have only finite automorphisms. Thus it remains to prove the algebraicity of both $\mathscr{M}(X, \tau)$ and $\mathscr{M}'(X, \tau)$.

Consider the Cartesian diagram

$$\begin{tikzcd} \mathscr{M}(X, \tau) \arrow[r] \arrow[d] & \mathscr{M}'(X, \tau) \arrow[d] \\ \mathfrak{M}(\mathcal{A}, \tau) \arrow[r, "(B.35)"] & \mathfrak{M}'(\mathcal{A}, \tau) \end{tikzcd} \tag{B.37}$$

where the vertical arrows are obtained by taking the associated punctured maps to $\mathcal{A}$, hence are strict. Since the bottom arrow is a strict closed embedding, the top arrow is representable. Thus it suffices to prove the algebraicity of $\mathscr{M}'(X, \tau)$.

Denote by $\mathscr{M}(\underline{X})$ the moduli of usual stable maps to $\underline{X}$, equipped with the canonical log structure of its domain universal curve. Similarly, let $\mathfrak{M}(\underline{\mathcal{A}})$ be the moduli of usual pre-stable maps to $\underline{\mathcal{A}}$ with its canonical log structure from its domain. We arrive at another Cartesian diagram

$$\begin{tikzcd} \mathscr{M}'(X, \tau) \arrow[r] \arrow[d] & \mathscr{M}(\underline{X}) \arrow[d] \\ \mathfrak{M}'(\mathcal{A}, \tau) \arrow[r] & \mathfrak{M}(\underline{\mathcal{A}}) \end{tikzcd}$$

where the two horizontal arrows are the tautological ones, and the right vertical arrow is induced by the underlying of $X \to \mathcal{A}$, hence is strict. Since all the three stacks $\mathscr{M}(\underline{X})$, $\mathfrak{M}(\underline{\mathcal{A}})$ and $\mathfrak{M}(\mathcal{A}, \tau)$ are algebraic and locally of finite type, we conclude that $\mathscr{M}'(X, \tau)$ is algebraic and locally of finite type. ☐

B.4.2. *Relative boundedness.* For a curve class $\beta \in H_2(X)$, consider the open substacks

$$\mathscr{M}(X, \tau, \beta) \subset \mathscr{M}(X, \tau), \qquad \mathscr{M}'(X, \tau, \beta) \subset \mathscr{M}'(X, \tau), \qquad \mathscr{M}(\underline{X}, \beta) \subset \mathscr{M}(\underline{X})$$

parameterizing punctured maps with curve class β. Consider the commutative triangle

$$\begin{tikzcd}
\mathscr{M}(X,\tau,\beta) \arrow[rr] \arrow[dr, "F"'] & & \mathscr{M}'(X,\tau,\beta) \arrow[dl, "F'"] \\
& \mathscr{M}(\underline{X},\beta) &
\end{tikzcd} \tag{B.38}$$

where the two arrows F, F' are the forgetful morphisms, and the horizontal arrow is a strict closed embedding induced by (B.37). For any strict morphism $W \to \mathscr{M}(\underline{X},\beta)$, we obtain pull-backs

$$F_W\colon \mathscr{M}(X,\tau,\beta)\times_{\mathscr{M}(\underline{X},\beta)} W \to W, \qquad F'_W\colon \mathscr{M}'(X,\tau,\beta)\times_{\mathscr{M}(\underline{X},\beta)} W \to W.$$

Similar to [20, §2.5], we introduce the following relative version of combinatorial finiteness which will be convenient in the situation of this paper. An absolute version with general targets can be found in [3, §3.3.1].

Definition B.30. The data (τ,β) is *F-combinatorially finite* (resp. F'-combinatorially finite) over W if the collection of tropical types of punctured maps in $\mathscr{M}(X,\tau,\beta)\times_{\mathscr{M}(\underline{X},\beta)}W$ (resp. $\mathscr{M}'(X,\tau,\beta)\times_{\mathscr{M}(\underline{X},\beta)} W$) is finite.

Proposition B.31. *Suppose that* (τ,β) *is* F-combinatorially finite *(resp. F'-combinatorially finite) over W. Then F_W (resp. F'_W) is of finite type.*

Proof. The proof is analogous to [3, Prop. 3.16], briefly summarized below. As the statement is local on W, we may assume that W is of finite type. We will only consider the case that (τ,β) is F-combinatorially finite, as the other case is similar.

We first stratify the stack $\mathscr{M}(X,\tau,\beta)\times_{\mathscr{M}(\underline{X},\beta)} W$, such that each stratum parameterizes punctured maps of the same tropical type. The combinatorial finiteness implies that there are only finitely many strata. Hence it remains to show that each stratum is of finite type.

Consider a type $\tau = (\mathbf{G},\boldsymbol{\sigma},\mathbf{c})$ given by a stratum $\mathscr{M}_\tau \subset \mathscr{M}(X,\tau,\beta)\times_{\mathscr{M}(\underline{X},\beta)} W$. The image of $\mathscr{M}_\tau$ in W necessarily factors through a stratum $W_\tau \subset W$ determined by the data $(\mathbf{G},\boldsymbol{\sigma})$ imposed on the level of underlying maps. It remains to show that punctured maps lifting the underlying map of W_τ and having contact order $\mathbf{c}$ form a bounded family. To achieve this, we may first follow [3, Prop. 3.16] to parameterizes all punctured curves with the underlying pre-stable curve given by the family over W_τ. Then similar to [24, §3.2] and [16, §5.4], one may further construct a finite type scheme parameterizing all possible punctured maps from these punctured curves. This leads to the boundedness of $\mathscr{M}_\tau$, hence the finite type property of F_W. ☐

B.4.3. *Valuative criterion.*

Proposition B.32. *The morphism F' in* (B.38) *satisfies the valuative criterion. More precisely, for any discrete valuation ring R with the quotient field K and the maximal ideal $\mathfrak{m}$, consider a commutative diagram of solid arrows of underlying stacks*

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & \underline{\mathscr{M}'(X,\tau,\beta)} \arrow[d, "\underline{F'}"] \\
\operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \mathscr{M}(\underline{X},\beta)
\end{tikzcd}$$

Then, possibly after replacing R by a finite extension of DVRs, and K by the induced finite field extension, there exists a unique dashed arrow making the above diagram commutative. The same property holds for F.

Proof. As the horizontal arrow in (B.38) is a strict closed embedding, it suffices to prove the statement for F'. For X a log scheme with a Zariski log structure, the statement is proved in [3, Thm. 3.18]. While in our situation we allow the target X and domain curves to be Deligne-Mumford stacks, the log structure on the target X is only of DF1, the simplest possible Zariski log structure. Thus, the proof is analogous to [3, Thm. 3.18]. We briefly summarize the idea below.

Denote by $\underline{f}\colon \underline{C} \to \underline{X}$ the stable map over $\operatorname{Spec} R$ given by the bottom arrow, and $f_\eta\colon C^\circ_\eta \to X$ the basic stable punctured map over η given by the top arrow with $\underline{\eta} = \operatorname{Spec} K$. The commutativity of the above diagram means that the underlying map of f_η is $\underline{f}|_\eta$. The goal is to extend f_η to a punctured map $f\colon C \to X$ over S with $\underline{S} = \operatorname{Spec} R$ such that the underlying of f is $\underline{f}$, and $f|_{\underline{\eta}} = f_\eta$.

The first step is to show that the central fiber $f|_{\underline{s}}$ if exists, its tropical type is uniquely determined by f_η and $\underline{f}$. Note that given the underlying map $\underline{f}$, to determined the tropical type of $f|_{\underline{s}}$, it suffices to determine the contact orders along nodes. Thus we may apply [16, §6.2] and [24, §4.1] to obtain the tropical type of $f|_{\underline{s}}$.

In the second step, we construct a unique punctured curve $C^\circ \to S$ as the domain curve. Away from the punctures, the construction of C° is identical to [24, §4.2] using the tropical type from the first step. Furthermore as explained in the proof of [3, Thm. 3.18], the tropical type uniquely determines the log structure along the punctures.

Finally, one can extend f_η to f over $\operatorname{Spec} R$ following [16, §6.3] and [24, §4.3]. $\square$

APPENDIX C. LOGARITHMIC ALIGNMENTS

We recall logarithmic alignments introduced in [1], and establish some basic properties. The results of this section are used crucially in §6 for comparing virtual cycles.

C.1. **Alignments of locally free log structures.**

C.1.1. *Aligned log structures.* Recall from [1, §8.1] that an *aligned log structure* on a scheme $\underline{S}$ is a locally free log structure $\mathcal{M}_S$, together with a sheaf of finite subsets $\overline{\mathbf{A}}_S \subset \overline{\mathcal{M}}_S$ such that for each geometric point $s \in S$ the fiber $\overline{\mathcal{M}}_S|_s \cong \mathbb{N}^m$ has a basis $\{e_1, e_2, \cdots, e_m\}$ and

$$\overline{\mathbf{A}}_S|_s = \{0, e_1, e_1 + e_2, e_1 + e_2 + e_3, \cdots, e_1 + \cdots + e_m\}. \tag{C.1}$$

Denote by $\mathcal{T}_{\log}$ the stack of aligned log structures. By [1, Proposition 8.1.2], $\mathcal{T}_{\log}$ is an algebraic stack. We view $\mathcal{T}_{\log}$ as a log stack with its universal aligned log structure $\mathcal{M}_{\log}$ and the universal subset $\overline{\mathbf{A}}_{\log} \subset \overline{\mathcal{M}}_{\log}$. Then $\mathcal{T}_{\log}$ is smooth and log smooth by [1, Proposition 8.2.2 and 8.3.1].

We introduce three types of alignments to be applied to log GLSM.

C.1.2. *The universal alignment.* Let $\overline{\mathcal{M}}^a_{\log} \subset \overline{\mathcal{M}}_{\log}$ be the free submonoid generated by $\overline{\mathbf{A}}_{\log}$. Denote by $\mathcal{M}^a_{\log} = \overline{\mathcal{M}}^a_{\log} \times_{\overline{\mathcal{M}}_{\log}} \mathcal{M}_{\log} \subset \mathcal{M}_{\log}$ the corresponding sub-log structure, and by $\mathcal{T}^a_{\log} = (\underline{\mathcal{T}}_{\log}, \mathcal{M}^a_{\log})$ the log stack.

Let $S \to \mathcal{T}_{\log}$ be a strict morphism given by an aligned log structure $(\mathcal{M}_S, \overline{\mathbf{A}}_S)$ as in §C.1.1. Then fiberwise the non-zero elements in $\overline{\mathbf{A}}_S$ form a set of generators of the pull-back log structure $\overline{\mathcal{M}}^a_S = \mathcal{M}^a_{\log}|_{\underline{S}}$. By (C.1), we observe that $\overline{\mathcal{M}}^a_S$, hence $\mathcal{M}^a_{\log}$ is locally free.

Let $\mathrm{Log}^{fr} \subset \mathrm{Log}$ be the open substack parameterizing locally free log structures. Then we obtain a composition, called the *universal alignment*:

$$\text{align}\colon \mathcal{T}_{\log} \xrightarrow{\mathcal{M}^a_{\mathcal{T}_{\log}} \subset \mathcal{M}_{\mathcal{T}_{\log}}} \mathcal{T}^a_{\log} \xrightarrow{\text{strict}} \mathrm{Log}^{fr}\,. \tag{C.2}$$

C.1.3. *Truncated alignments.* Let $\mathrm{Log}^{fr}_{\leq m} \subset \mathrm{Log}^{fr}$ be the open dense substack over which the fibers of $\overline{\mathcal{M}}_{\mathrm{Log}^{fr}}$ have rank $\leq m$. Let $\mathrm{Log}^{fr}_{=m} \subset \mathrm{Log}^{fr}_{\leq m}$ be the reduced strict closed substack over which the fibers of $\overline{\mathcal{M}}_{\mathrm{Log}^{fr}_{\leq m}}$ have rank precisely m.

The morphism (C.2) restricts to *truncated alignments*:

$$\mathrm{align}_\bullet := \mathrm{align}|_{\mathcal{T}_\bullet} \colon \mathcal{T}_\bullet := \mathcal{T}_{\log} \times_{\mathrm{Log}^{fr}} \mathrm{Log}^{fr}_\bullet \longrightarrow \mathrm{Log}^{fr}_\bullet \tag{C.3}$$

where $\bullet$ represents $\leq m$ or $= m$. For later use, denote by $\mathcal{M}^a_\bullet := \mathrm{align}^*_\bullet \mathcal{M}_{\mathrm{Log}^{fr}_\bullet}$.

We call $\mathcal{T}_{\leq m}$ resp. $\mathcal{T}_{=m}$ the *stack of aligned log structures of length* $\leq m$ *(resp.* $= m$*)*. We also obtain the universal subsets $\overline{\mathbf{A}}_{\leq m} = \overline{\mathbf{A}}_{\log}|_{\mathcal{T}_{\leq m}}$ and $\overline{\mathbf{A}}_{=m} = \overline{\mathbf{A}}_{\log}|_{\mathcal{T}_{=m}}$.

C.1.4. *Labeled alignments.* We are also interested in the case where all elements in (C.1) are labeled. Consider $\mathcal{A}^m \cong \prod_{i\in[m]} \mathcal{A}_i$ with the m copies of $\mathcal{A}_i \cong \mathcal{A}$ labeled by the set $[m] := \{1, 2, \cdots, m\}$. Taking base change along the strict étale surjective morphism $\mathcal{A}^m \to \mathrm{Log}^{fr}_{\leq m}$, we obtain the $[m]$*-labeled alignment*:

$$\mathrm{align}_{[\leq m]} := \mathrm{align}_{\leq m} \times_{\mathrm{Log}^{fr}_{\leq m}} \mathcal{A}^m \colon \mathcal{T}_{[\leq m]} := \mathcal{T}_{\leq m} \times_{\mathrm{Log}^{fr}_{\leq m}} \mathcal{A}^m \to \mathcal{A}^m. \tag{C.4}$$

For any strict morphism $S \to \mathcal{T}_{[\leq m]}$ induced by an aligned log structure $(\mathcal{M}_S, \overline{\mathbf{A}}_S)$, non-zero sections in $\overline{\mathbf{A}}_S$ are labeled by $[m]$. Pulling back $\mathrm{align}_{[\leq m]}$ along the strict closed imbedding $\infty^m_{\mathcal{A}} \to \mathcal{A}^m$, we obtain the $[m]$*-labeled alignment of length* m:

$$\mathrm{align}_{[m]} \colon \mathcal{T}_{[m]} := \mathcal{T}_{[\leq m]} \times_{\mathcal{A}^m} \infty^m_{\mathcal{A}} \to \infty^m_{\mathcal{A}}. \tag{C.5}$$

C.1.5. *Properties of logarithmic alignments.*

Proposition C.1. *The morphism* (C.4) *is birational, projective and log étale.*

Proof. We first check the log étaleness. By [1, §8.3], $\mathcal{T}_{[\leq m]}$ admits a cover given by strict étale morphisms $\mathcal{A}^n \to \mathcal{T}_{[\leq m]}$ for $n \leq m$. It suffices to show that the composition

$$\begin{tikzcd} \mathcal{A}^n \arrow[r] & \mathcal{T}_{[\leq m]} \arrow[r, "\mathrm{align}_{[\leq m]}"] & \mathcal{A}^m \end{tikzcd}$$

is log étale. However, this is a morphism between two Artin fans, whose log étaleness follows from the local criterion [27, Theorem (3.5)].

The birationality follows from the fact that $\mathrm{align}_{[\leq m]}|_{\mathcal{T}_{[\leq 1]}} = \mathrm{align}_{[\leq 1]}$ is the identity on the open dense substack $\mathcal{T}_{[\leq 1]} \subset \mathcal{T}_{[\leq m]}$.

It remains to show the projectivity. In the following, we will show that $\mathcal{T}_{[\leq m]}$ is the moduli $X^{[m]}$ of degree m stable configurations of points in the pair $X = (\mathcal{A}, \infty_{\mathcal{A}})$ with $\mathrm{align}_{[\leq m]}$ the evaluation given by the m sections labeled by $[m]$, see [4, Definition 1.5.2]. Then the projectivity follows from [4, Proposition 1.5.4].

We view $X^{[m]}$ as a log stack with the canonical log structure given by its universal expansions. For any strict morphism $S \to X^{[m]}$, denote by $S[m] \to S$ the family of expansions. Over each geometric point $s \in S$, we have the fiber of length $k \leq m$ (see [4, Convention 1.4.1])

$$S[m]_s = X \cup_{\infty = \infty_-} P_1 \cup_{\infty_+ = \infty_-} \cdots \cup_{\infty_+ = \infty_-} P_k.$$

The fiber $\overline{\mathcal{M}}_S|_s \cong \mathbb{N}^k$ is a free monoid with generators $e_1, \cdots, e_k$ such that e_i corresponds to the smoothing parameter of the node given by $\infty_- \subset P_i$.

Recall that the family of stable expansions $S[m] \to S$ admits m sections $\epsilon_k \colon S \to S[m]$ for $k = 1, \cdots, m$. Each P_i has a unique open dense point, which contains at least one section by stability [4, Definition 1.5.1]. For the kth section $\epsilon_k \colon S \to S[m]$, we define an element $\delta_{k,s} = e_1 + e_2 + \cdots + e_{k_i}$ if the fiber $\epsilon_k|_s$ lands in P_{k_i}. This defines a set $\overline{\mathbf{A}}_{S,s} := \{\delta_{k,s} \mid k \in [m]\} \cup \{0\}$ of length $\leq m$ since sections can intersect. Observe that the fiberwise defined $\overline{\mathbf{A}}_{S,s}$ glues to a sheaf of totally ordered subsets $\overline{\mathbf{A}}_S \subset \overline{\mathcal{M}}_S$ labeled by $[m]$. This defines a morphism $S \to \mathcal{T}_{[\leq m]}$, hence $X^{[m]} \to \mathcal{T}_{[\leq m]}$.

To obtain the inverse $\mathcal{T}_{[\leq m]} \to X^{[m]}$, note that $\mathcal{T}_{[\leq m]}$ carries a natural family of expansions of length $\leq m$ by [1, Proposition 8.1.2]. The log structure $\mathcal{M}_{[\leq m]}$ is the canonical log structure of the expansions. For a strict morphism $S \to \mathcal{T}_{[\leq m]}$, let $S[m] \to S$ be the family of expansions pulled back from $\mathcal{T}_{[\leq m]}$. The section $\epsilon_k \colon S \to S[m]$ is constructed using the section in $\overline{\mathbf{A}}_{[\leq m]}$ labeled by $k \in [m]$ by reversing the above construction.

This completes the proof. $\square$

Corollary C.2. *The morphism* (C.2) *is proper, representable, birational, and log étale.*

Proof. We may check this on a strict étale cover. Note that the collection of compositions $\mathcal{A}^m \to \mathrm{Log}^{fr}_{\leq m} \to \mathrm{Log}^{fr}$ of strict étale morphisms for all m form an étale cover of Log^{fr}. Thus the statement follows from Proposition C.1. $\square$

Corollary C.3. *The morphism* $\mathrm{align}_{[m]}$ *is projective and log étale. It is birational iff* $m = 1$*.*

Proof. This follows from Proposition C.1 by taking the base change of (C.2) along $\infty^m_{\mathcal{A}} \to \mathcal{A}^m$. For birationality, note that $\mathrm{align}_{[1]}$ is an isomorphism by the definition of aligned log structures. $\square$

C.2. **General alignments.** In general, the log structures that we will need to deal with are rarely locally free. Next generalize the alignments from §C.1, as required for §6.

C.2.1. *The* $\overline{\mathbf{A}}$*-alignments.* Let S be an fs log stack with a sheaf of finite subsets $\overline{\mathbf{A}} \subset \overline{\mathcal{M}}_S$ containing 0. A morphism $h \colon T \to S$ is called an *alignment* of $\overline{\mathbf{A}}$ if the image of

$$h^{-1}\overline{\mathbf{A}} \to f^{-1}\overline{\mathcal{M}}_S \to \overline{\mathcal{M}}_T$$

is geometric fiberwise totally ordered with respect to $\preccurlyeq_{\overline{\mathcal{M}}_T}$. Note that if $T \to S$ is an alignment of $\overline{\mathbf{A}}$, then any composition $T' \to T \to S$ from an fs log scheme T' is also an alignment of $\overline{\mathbf{A}}$.

Let $\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S$ be the sheaf of locally free monoids over $\underline{S}$ whose fiberwise generators are labeled by non-zero elements of $\overline{\mathbf{A}}$. The inclusion $\overline{\mathbf{A}} \subset \overline{\mathcal{M}}_S$ induces a morphism $\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S \to \overline{\mathcal{M}}_S$ hence a log structure $\mathcal{M}^{\overline{\mathbf{A}}}_S = \mathcal{M}_S \times_{\overline{\mathcal{M}}_S} \overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S$ over $\underline{S}$ with the structure morphism defined by the composition $\mathcal{M}^{\overline{\mathbf{A}}}_S \to \mathcal{M}_S \to \mathcal{O}_S$. Consider the composition

$$\begin{tikzcd} S \arrow[r, "\mathcal{M}_S \leftarrow \mathcal{M}^{\overline{\mathbf{A}}}_S"] & (\underline{S}, \mathcal{M}^{\overline{\mathbf{A}}}_S) \arrow[r, "\mathcal{M}^{\overline{\mathbf{A}}}_S"] & \mathrm{Log}^{fr} \end{tikzcd}$$

Pulling back (C.2), we obtain the projection

$$\mathrm{align}_{\overline{\mathbf{A}}} \colon \mathcal{T}_{\overline{\mathbf{A}}} := \mathcal{T}_{\log} \times_{\mathrm{Log}^{fr}} S \to S, \tag{C.6}$$

called the $\overline{\mathbf{A}}$*-alignment.*

Denote by $\mathcal{M}^a_{\mathcal{T}_{\overline{\mathbf{A}}}}$ the pull-back $\mathcal{M}^a_{\log}|_{\mathcal{T}_{\overline{\mathbf{A}}}} \cong \mathcal{M}^{\overline{\mathbf{A}}}_S|_{\mathcal{T}_{\overline{\mathbf{A}}}}$. The sheaf $\overline{\mathbf{A}}_{\log}|_{\mathcal{T}_{\overline{\mathbf{A}}}}$ is fiberwise given by the set of generators of $\overline{\mathcal{M}}^a_{\mathcal{T}_{\overline{\mathbf{A}}}}$, hence we have an identification $\overline{\mathbf{A}}_{\log}|_{\mathcal{T}_{\overline{\mathbf{A}}}} = \overline{\mathbf{A}}|_{\mathcal{T}_{\overline{\mathbf{A}}}}$. Therefore, the image of the composition

$$\overline{\mathbf{A}}|_{\mathcal{T}_{\overline{\mathbf{A}}}} = \overline{\mathbf{A}}_{\log}|_{\mathcal{T}_{\overline{\mathbf{A}}}} \to \overline{\mathcal{M}}^a_{\mathcal{T}_{\overline{\mathbf{A}}}} \to \overline{\mathcal{M}}_{\mathcal{T}_{\overline{\mathbf{A}}}}$$

is totally ordered with respect to $\preccurlyeq_{\overline{\mathcal{M}}_{\mathcal{T}_{\overline{\mathbf{A}}}}}$. This shows that the projection $\mathcal{T}_{\overline{\mathbf{A}}} \to S$ is an alignment of $\overline{\mathbf{A}}$. Indeed $\mathcal{T}_{\overline{\mathbf{A}}}$ is the universal alignment:

Proposition C.4. *For any* $h \colon T \to S$ *and an alignment of* $\overline{\mathbf{A}}$*, there is a unique factorization*

$$\begin{tikzcd} T \arrow[r, "h'"] \arrow[dr, bend right, "h"'] & \mathcal{T}_{\overline{\mathbf{A}}} \arrow[d, "\mathrm{align}_{\overline{\mathbf{A}}}"] \\ & S. \end{tikzcd}$$

Proof. Consider the sequence of morphisms of log structures $\mathcal{M}^{\overline{\mathbf{A}}}_S|_{\underline{T}} \longrightarrow \mathcal{M}_S|_{\underline{T}} \xrightarrow{h^\flat} \mathcal{M}_T$. We construct a sub-sheaf of monoids $\overline{\mathcal{M}}' \subset (\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S|_T)^{gp}$ over $\underline{T}$ as follows.

For a geometric point $t \in \underline{T}$, let $\{\delta_1, \cdots, \delta_m\}$ be the set of generators of $\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S|_t$. Denote by $\delta'_i \in \overline{\mathcal{M}}_T|_t$ the image of δ_i. After reordering, we may assume that

$$\delta'_1 \preccurlyeq_{\overline{\mathcal{M}}_T} \delta'_2 \preccurlyeq_{\overline{\mathcal{M}}_T} \cdots \preccurlyeq_{\overline{\mathcal{M}}_T} \delta'_m.$$

Define the fiber $\overline{\mathcal{M}}'|_t \subset (\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S|_t)^{gp}$ to be the free monoid generated by

$$\delta_1, \delta_2 - \delta_1, \cdots, \delta_m - \delta_{m-1}.$$

This fiberwise construction glues to a subsheaf of monoids $\overline{\mathcal{M}}' \subset (\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S|_T)^{gp}$.

The fact that h is an alignment of $\overline{\mathbf{A}}$ implies that $(\delta'_{i+1} - \delta'_i) \in \overline{\mathcal{M}}_T$. Thus the composition $\overline{\mathcal{M}}' \subset (\overline{\mathcal{M}}^{\overline{\mathbf{A}}}_S|_{\underline{T}})^{gp} \to \overline{\mathcal{M}}^{gp}_T$ factors through $\overline{\mathcal{M}}' \to \overline{\mathcal{M}}_T$. This defines a log structure $\mathcal{M}' := \mathcal{M}_T \times_{\overline{\mathcal{M}}_T} \overline{\mathcal{M}}'$ with the structure morphism given by $\mathcal{M}' \to \mathcal{M}_T \to \mathcal{O}_T$. In particular, we obtain a morphism of log schemes $T \to (\underline{T}, \mathcal{M}')$. By construction, the pair $(\overline{\mathbf{A}}_T, \mathcal{M}'_T)$ defines an aligned log structure on $\underline{T}$, inducing a strict morphism $(\underline{T}, \mathcal{M}') \to \mathcal{T}_{\log}$. This leads to a commutative diagram

$$\begin{tikzcd}
T \arrow[r, "h"] \arrow[d] & S \arrow[d] \\
\mathcal{T}_{\log} \arrow[r, "\mathrm{align}"] & \mathrm{Log}^{fr}
\end{tikzcd}$$

with the left vertical arrow given by $T \to (\underline{T}, \mathcal{M}') \to \mathcal{T}_{\log}$. This leads to a unique factorization h' as in the statement, finishing the proof. ☐

C.2.2. $\mathrm{align}_{\overline{\mathbf{A}}}$ *in the labeled case.* Consider a sheaf of finite subsets $\overline{\mathbf{A}} \subset \overline{\mathcal{M}}_S$ over S as in §C.2.1. An $[m]$-*labeling* of $\overline{\mathbf{A}}$ is a strict morphism $(\underline{S}, \mathcal{M}^{\overline{\mathbf{A}}}_S) \to \mathcal{A}^m \cong \prod_{i \in [m]} \mathcal{A}_i$ with the m copies of $\mathcal{A}_i \cong \mathcal{A}$ labeled by $[m]$. In particular each non-zero (local) section of $\overline{\mathbf{A}}$ is uniquely labeled by an element of $[m]$.

In case $(\underline{S}, \mathcal{M}^{\overline{\mathbf{A}}}_S) \to \mathcal{A}^m$ factors through $\infty^m_{\mathcal{A}}$, note that $\overline{\mathbf{A}} \setminus \{0\}$, as the sheaf of generators of $\overline{\mathcal{M}}_{\infty^m_{\mathcal{A}}}|_S$, is the constant sheaf with m elements.

Proposition C.5. *Suppose* $\overline{\mathbf{A}}$ *is* $[m]$*-labeled. Then* $\mathrm{align}_{\overline{\mathbf{A}}}$ *in* (C.6) *is projective and log étale.*

Proof. This follows from Corollary C.3 and the following Cartesian squares in the fs category:

$$\begin{tikzcd}
\mathcal{T}_{\overline{\mathbf{A}}} \arrow[r] \arrow[d, "\mathrm{align}_{\overline{\mathbf{A}}}"'] & \mathcal{T}_{[\leq m]} \arrow[r] \arrow[d, "\mathrm{align}_{[\leq m]}"'] & \mathcal{T}_{\log} \arrow[d, "\mathrm{align}"] \\
S \arrow[r] & \mathcal{A}^m \arrow[r] & \mathrm{Log}^{fr}
\end{tikzcd}$$

☐

C.2.3. *Order removing.* Suppose the sheaf of subsets $\overline{\mathbf{A}} \subset \overline{\mathcal{M}}_S$ over S is $[m]$-labeled. Consider the subset $[m'] \subset [m]$ for some $m' \leq m$. This induces a subsheaf $\overline{\mathbf{A}}' \subset \overline{\mathbf{A}}$ by taking elements labeled by $[m']$. Since any alignment of $\overline{\mathbf{A}}$ is automatically an alignment of $\overline{\mathbf{A}}'$, by Proposition C.4 we obtain a natural morphism over S:

$$\mathrm{align}_{\overline{\mathbf{A}} \supset \overline{\mathbf{A}}'} \colon \mathcal{T}_{\overline{\mathbf{A}}} \to \mathcal{T}_{\overline{\mathbf{A}}'}. \tag{C.7}$$

By Proposition C.5, the morphism $\mathrm{align}_{\overline{\mathbf{A}} \supset \overline{\mathbf{A}}'}$ is projective over S.

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(Q. Chen) Department of Mathematics, Boston College, Chestnut Hill, MA 02467, U.S.A.
Email address: `qile.chen@bc.edu`

(F. Janda) Department of Mathematics, University of Illinois Urbana–Champaign, Urbana, IL 61801, U.S.A.
Email address: `fjanda@illinois.edu`

(Y. Ruan) Institute for Advanced Study in Mathematics, Zhejiang University, Hangzhou, China
Email address: `ruanyb@zju.edu.cn`