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RELATIONS IN THE TAUTOLOGICAL RING

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Abstract

The tautological ring of the moduli space of curves is the ring generated by algebraic cycles of geometric origin. While it has been well-studied, its structure is not yet completely understood : There is an explicit set of generators but the set of relations between them is still not known. Recently, A. Pixton has given a computationally well-tested conjectural description of this set.

This thesis contains an introduction to the moduli space of curves and its tautological ring, as well as three papers which provide evidence for the conjecture of Pixton. The first gives a proof that the elements of Pixton's set are actual relations. In cohomology, this result had already previously been obtained by Pandharipande-Pixton-Zvonkine with an at first sight very different method of proof.

The second paper notes that the method proof employed in the first paper can actually be interpreted as a careful study of the Givental-Teleman classification of cohomological field theories, like it is done in the work of Pandharipande-Pixton-Zvonkine, but in the context of a different example. The main part of the paper is a comparison of the relations obtained from cohomological field theories in the examples of the equivariant Gromov-Witten theory of projective spaces and of Witten's r-spin class.

The third paper shows that the method of producing tautological relations by studying the Givental-Teleman classification for *any* cohomological field theory will only yield relations inside Pixton's set. This shows that Pixton's relations have a universal character and gives further evidence for his conjecture.

Résumé

L'anneau tautologique de l'espace de modules des courbes algébriques est l'anneau engendré par des classes de cycles algébriques construites de façon géométrique. Tandis qu'il a fait l'objet de très nombreux travaux, sa structure n'est pas complètement connue. En effet, l'anneau tautologique possède une partie génératrice mais la totalité de ses relations n'est pas encore connue. Récemment, A. Pixton a donné une description conjecturale des relations en question et il est à noter que cette conjecture est computationnellement vérifiée dans de nombreux cas.

Cette thèse contient une introduction à l'espace de modules des courbes et à l'anneau tautologique ainsi que trois prépublications concernant les conjectures de Pixton. Dans la première prépublication, nous démontrerons que les éléments de l'ensemble de Pixton sont de vraies relations. En cohomologie, ce résultat a déjà été obtenu par Pandharipande-Pixton-Zvonkine par des méthodes à première vue très différentes.

Dans la deuxième prépublication, nous noterons que nos méthodes pour obtenir des relations sont essentiellement identiques aux méthodes utilisées par Pandharipande-Pixton-Zvonkine. Comme ces derniers, nous étudions la classification des théories cohomologiques des champs de Givental-Teleman à la différence près que nous considérerons des exemples différents. De plus, nous comparerons les relations obtenues par la théorie cohomologique des champs dans le cas de la théorie équivariante de Gromov-Witten de l'espace projectif avec celles obtenues dans le cas de la classe r-spin de Witten.

Dans la troisième prépublication, nous verrons que les relations obtenues grâce à la classification de Givental-Teleman sont contenues dans l'ensemble de Pixton et ce pour *toute* théorie cohomologique des champs. Les relations de Pixton ont par conséquent un charactère universel, ce qui tend à confirmer la conjecture du même nom.

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Part I
Introduction

1 Algebraic curves and their moduli

We give a short informal introduction to the moduli space of curves. For a reference on the subject refer for example to the book [24].

1.1 Smooth curves

Let us consider the problem of classifying smooth (connected) algebraic curves C of genus g up to isomorphism or equivalently classifying isomorphism classes of compact Riemann surfaces of genus g or, also equivalently, classifying complex structures on a compact, orientable surface of genus g.

For example we could try to find an assignment which maps any isomorphism class to a collection of complex numbers, which uniquely determine the isomorphism class. The image X of such an assignment in complex affine space would give us a geometric object such that every isomorphism class of algebraic curves of genus g corresponds to a point on X. We should better assume that the assignment varies nicely when deforming the curves so that X does not only record the cardinality of the set of isomorphism classes.

This notion is captured by the definition of a *fine* moduli scheme. A scheme M_g is a fine moduli space for the moduli problem of "smooth algebraic curves of genus g up to isomorphism" if there exists a universal family $C_g \to M_g$ of smooth algebraic curves of genus g over M_g such that for any base scheme S and any family $C \to S$ of smooth algebraic curves of genus g there exists a map $f : S \to M_g$ such that the family C is isomorphic to the pullback of the universal family C_g. Taking in this definition S to be a point, we see that the fiber of C_g over a point of M_g, which corresponds to an isomorphism class $[C]$ of algebraic curves, is isomorphic to C.

It turns out that such a moduli scheme cannot exist in general. The main problem is that there exist non-trivial families of algebraic curves such that each fiber is isomorphic to a fixed algebraic curve with a non-trivial automorphism group. On the other hand, if one does not restrict one's search of a fine moduli space to schemes, for $g \geq 2$ one can construct a fine moduli stack M_g, which is a smooth Deligne-Mumford stack (or also an (in general ineffective) orbifold). As a note to the reader unfamiliar with stacks or orbifolds we need to remark though that the fact that M_g is not a scheme will not play much of a role in this thesis. Being a Deligne-Mumford stack means that M_g is still quite close to being a scheme.

A first natural question about the moduli space M_g is its dimension. This was already answered in 19th century with a parameter count of Riemann: $\dim_{\mathbb{C}}(M_g) = 3g - 3$.

1.2 Compactification

While the moduli space M_g is smooth, it is not proper (compact): It is easy to construct families of algebraic curves over the projective line, which degenerate to a singular curve. In order to obtain a compact moduli space, an idea is to add points to the moduli space corresponding to some of these singular objects. For this it will first be useful to generalize the moduli problem in another direction:

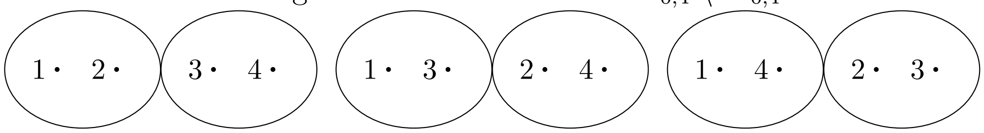
For $2g - 2 + n > 0$ there exists a smooth Deligne-Mumford stack $M_{g,n}$ which is a fine moduli stack for the problem of classifying isomorphism classes of smooth algebraic curves of genus g together with n pairwise disjoint markings, labeled (usually) by the set $\{1, \ldots, n\}$. It has complex dimension $3g - 3 + n$.

Next, for the compact moduli space we also need to consider nodal curves, i.e. algebraic curves such that each point is either smooth or étale locally looks like a coordinate cross $\{xy = 0\}$. Marked curves arise when we consider the normalization $\tilde{C}$ of C together with the preimages of the nodes. Then the normalization map $\tilde{C} \to C$ can be visualized as gluing the smooth (possibly disconnected) algebraic curve $\tilde{C}$ along marked points.

Now we can go to the compactification: For $2g - 2 + n > 0$ there is a smooth, proper Deligne-Mumford (fine) moduli stack $\overline{M}_{g,n}$ of stable, connected nodal curves of arithmetic genus g together with n labeled markings, which are pairwise disjoint and also disjoint from the nodes. Stability means that for every nodal curve C and any connected component D of the normalization $\tilde{C}$ the condition $2g_D - 2 + n_D > 0$ should hold, where g_D is the genus of the component and n_D is the number of preimages of markings and nodes on D.

For low values of g and n the space $\overline{M}_{g,n}$ can be made very concrete. For example $\overline{M}_{0,3}$ is a point because for 3 distinct points on the Riemann sphere there exists exactly one linear fractional transformation sending them to 0,

Figure 1: The elements of $\overline{M}_{0,4} \setminus M_{0,4}$



1 and ∞. The moduli space $\overline{M}_{0,4}$ is isomorphic to $\mathbb{P}^1$: The part where all four points are distinct is isomorphic to $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ since we can map the first three points to 0, 1 and ∞ and consider the image of the fourth point. The three missing points of $\mathbb{P}^1$ correspond to the three marked nodal curves illustrated in Figure 1.

By the universal property there exists a universal family $\overline{C}_{g,n} \to \overline{M}_{g,n}$ of stable curves. Here the space $\overline{C}_{g,n}$ is called the *universal curve*. The markings define n sections for the map $\overline{C}_{g,n} \to \overline{M}_{g,n}$: The ith section sends the class of a marked curve $(C, p_1, \ldots, p_n)$ to the point p_i on the fiber over $(C, p_1, \ldots, p_n)$.

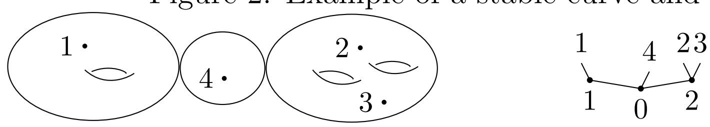
1.3 Boundary

The moduli space $M_{g,n}$ is an open subset inside $\overline{M}_{g,n}$ and it is interesting to study its boundary $\overline{M}_{g,n} \setminus M_{g,n}$. It turns out to have a very explicit description in terms of moduli spaces of curves with smaller genus or markings.

Consider any stable nodal curve C together with the map f from its normalization $\tilde{C}$. From here we can attach discrete data to the moduli point $[C] \in \overline{M}_{g,n}$ by recording for all connected components of $\tilde{C}$, the genus, what markings lie on it and how the components are glued together. This data can be collected into what is called a *dual graph*: We draw a vertex for any connected component of $\tilde{C}$ and decorate it by the genus. For every node of C we draw an edge connecting the vertices corresponding to the components containing the two preimages of the node. Therefore there can be multiple edges or self-edges. Finally for every marking of C, we draw a labeled half-edge. A simple example is illustrated in Figure 2.

It is often useful to think of each edge of a dual graph to be glued from two half-edges. For example, in many formulas we will use the number of *automorphisms of the dual graph* thought of in this way. For the dual graph with only one vertex and one loop there is exactly one non-trivial automorphism, which exchanges the two half-edges of the loop.

Figure 2: Example of a stable curve and its dual graph



The locus in $\overline{M}_{g,n}$ of curves corresponding to a fixed dual graph is locally closed and therefore these loci define a stratification of $\overline{M}_{g,n}$.

The closure of any stratum is essentially a product of smaller moduli spaces of curves. More precisely, given a dual graph Γ and for every vertex v of Γ an element $(C, p_1, \ldots, p_{n_v}) \in \overline{M}_{g_v,n_v}$, where g_v is the genus at v and n_v is the number of half-edges (corresponding to nodes and markings) at v, we can glue the stable curves together along the markings. This works well in families and we therefore obtain a map

$$\prod_v \overline{M}_{g_v,n_v} \to \overline{M}_{g,n}.$$

This map is finite with degree equal to the number of automorphisms of Γ.

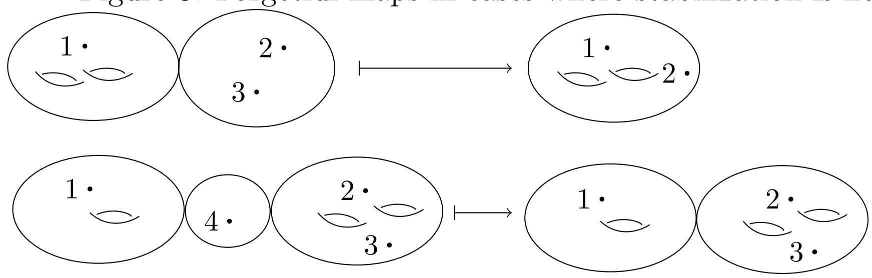
Apart from the one-vertex no-loops dual graph corresponding to smooth curves, the most simple dual graphs correspond to *boundary divisors* of $\overline{M}_{g,n}$. There are two possibilities: Either the dual graph has one vertex and one loop or it has two vertices connected by an edge but no loops. The first case corresponds to the closure of the locus of irreducible curves with a node. In the second case there are several possibilities how the genus and markings can be distributed among the two vertices. A generic element of the second locus is a reducible curve obtained from two smooth curves with genus and markings according to the graph by gluing along two additional markings.

Using dual graphs we can also define well-known partial compactifications of $M_{g,n}$ in $\overline{M}_{g,n}$. There is a sequence

$$M_{g,n} \subset M_{g,n}^{rt} \subset M_{g,n}^{ct} \subset \overline{M}_{g,n}.$$

Here, “rt” and “ct” are abbreviations for “rational tails” and “compact type”, respectively. A stable curve C lies in $M_{g,n}^{rt}$ if the dual graph of C is a tree, and one vertex has genus g while all the others have genus zero, i.e. are rational. A stable curve C is of compact type if its dual graph is a tree or equivalently when the Jacobian of C is compact.

Figure 3: Forgetful maps in cases where stabilization is necessary



1.4 Tautological maps

There is a big collection of canonical maps between moduli spaces of stable curves, which are called tautological maps.

As we have already seen there are algebraic maps obtained from gluing curves according to a dual graph. For divisors the *gluing maps* take the form

$$\begin{aligned} \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to & \overline{M}_{g_1+g_2,n_1+n_2}, \\ \overline{M}_{g-1,n+2} \to & \overline{M}_{g,n}, \end{aligned}$$

where in the first case the last markings of two stable curves are glued together and in the second case the two last markings of a stable curve are glued together. Any more general gluing map can be constructed by composition from these two basic types of gluing maps.

There is a second type of tautological map, called *forgetful map*, of the form

$$\overline{M}_{g,n+1} \to \overline{M}_{g,n},$$

which for most points of $\overline{M}_{g,n+1}$ maps a marked curve to the same curve but with the last marking forgotten. This definition does not work in the case the marking that needs to be forgotten lies on a Riemann sphere with only 3 special points, i.e. nodes or markings, since after forgetting the marking the curve would no longer satisfy the stability condition. To fix the definition, the curve has to be stabilized after the point has been forgotten, i.e. the unstable components need to be contracted, as illustrated in Figure 1.4.

2 Definition of the tautological ring

2.1 Tautological classes

As Mumford remarks in [37]:

> "Whenever a variety or topological space is defined by some universal property, one expects that by virtue of its defining property, it possesses certain cohomology classes called tautological classes."

His main motivating example is the Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$. In this case it is natural to consider the tautological bundle E of $\mathrm{Gr}(k,n)$, which is of rank k. The Chern classes of E define classes in the cohomology and Chow ring and should be counted as tautological. It happens that for the Grassmannian that these classes are already enough to generate the complete cohomology and Chow ring.

With [37], Mumford has started the study of tautological classes in the rational[1] Chow and cohomology ring of $\overline{M}_g$. Here the universal object is the universal curve $\overline{C}_g$ over $\overline{M}_g$. Since $\overline{C}_g$ is not a vector bundle over $\overline{M}_g$ we cannot directly take Chern classes, however there are natural bundles over $\overline{C}_g$ we can use. The tautological classes that Mumford considered are known as the κ- and λ-classes. We want to define them on $\overline{M}_{g,n}$ whereas Mumford had only considered $\overline{M}_g$.

Over $\overline{C}_{g,n}$ we can consider the dualizing sheaf ω relative to the universal family $\pi : \overline{C}_{g,n} \to \overline{M}_{g,n}$. We can also consider the log relative dualizing sheaf ω_{log}, which is defined from ω by twisting along the divisors $D_1, \dots, D_n$ defined as the images of the n sections $s_1, \dots, s_n$ of π:

$$\omega_{\mathrm{log}} = \omega(D_1 + \cdots + D_n).$$

We can now define

$$\begin{aligned}
\kappa_d &:= \pi_*(c_1(\omega_{\mathrm{log}})^{d+1}) && \in A^d(\overline{M}_{g,n}),\\
\psi_i &:= s_i^*(c_1(\omega)) && \in A^1(\overline{M}_{g,n}),\\
\lambda_d &:= c_d(\pi_*\omega) && \in A^d(\overline{M}_{g,n}).
\end{aligned}$$

1. This is needed because of the stacky structure of $\overline{M}_{g,n}$. We will suppress the $\mathbb{Q}$-coefficients in the notation.

The definition of κ_d that we use has first appeared in [1]. Closely related classes can be defined by replacing $\omega_{\log}$ by ω in the definition. Also notice that $\kappa_0 = 2g - 2 + n$ because that is the degree of $\omega_{\log}$. The rank g bundle $\mathbb{E} := c_d(\pi_* \omega)$ in the definition of the λ-classes is known as the *Hodge bundle.*

Mumford already realized that these tautological classes are not independent. As an application of the Grothendieck-Riemann-Roch formula applied to the morphism π he computed the total Chern character of the Hodge bundle:

$$ch(\mathbb{E}) = 1 + \sum_{d=0}^{\infty} \frac{(-1)^d B_{d+1}}{(d+1)!} \left(-\kappa_d + \sum_{i=1}^{n} \psi_i^d - \sum_{\Delta} \frac{1}{|\operatorname{Aut}(\Delta)|} i_{\Delta *} \frac{\psi_a^d + \psi_b^d}{\psi_a + \psi_b} \right)$$

Here the last sum is over isomorphism classes of dual graphs Δ of boundary divisors, i_Δ is the corresponding gluing map and ψ_a and ψ_b stand for the ψ-classes at the two markings which are glued together. The B_d are Bernoulli numbers, which are defined by the generating series

$$\sum_{d=0}^{\infty} \frac{B_d}{d!} x^d = \frac{x}{e^x - 1}.$$

Because the Chern characters and the total Chern class are in general related via

$$\sum_{d=0}^{\infty} c_d(\mathbb{E}) = \exp\left(\sum_{d=1}^{\infty} (-1)^{d-1} (d-1)! \cdot ch_d(\mathbb{E}) \right),$$

Mumford's formula implies that the λ-classes can be expressed in terms of κ-, ψ- and boundary classes.

Boundary classes as in Mumford's formula which are obtained from tautological classes via push-forward along the gluing maps are also considered as tautological.

2.2 The tautological ring

A natural question to ask is whether as for the Grassmannian all the Chow and cohomology classes of $\overline{M}_{g,n}$ can be expressed in terms of tautological classes. Unfortunately this is not true, there exist non-tautological classes [20, 15].

Still we can define a subring of the Chow ring of $\overline{M}_{g,n}$ as the ring generated by all κ-, ψ-, λ-classes and classes obtained from them by push-forward along the gluing maps. This subring

$$R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$$

is called the *tautological ring*. A very compact definition is given in [14]:

The system of tautological rings of the moduli space of curves is the smallest system of subrings $R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$ closed under push-forward by the tautological gluing and forgetful maps.

Since rings and subrings include an identity element, this definition at least does not produce the empty set. Furthermore, by using the self-intersection formula and the deformation theory of stable curves, the class ψ_i can be written as

$$\pi_*(-j_*(1)^2),$$

where here $\pi : \overline{M}_{g,n+1} \to \overline{M}_{g,n}$ is the forgetful map, $j : \overline{M}_{g,n} \times \overline{M}_{0,3} \to \overline{M}_{g,n+1}$ is a gluing map and 1 is the identity. We still need to say that for j we put the markings i and $n+1$ on the rational component and all other markings at the genus g component. From the ψ-classes we can also construct κ-classes via

$$\pi_*(\psi_{n+1}^{d+1}) = \kappa_d, \tag{1}$$

and λ-classes can then be obtained via Mumford's formula.

In fact, there is a (finite) generating set for $R^*(\overline{M}_{g,n})$ as first explicitly written down in [20]. The generators correspond to dual graphs together with for each vertex v the decoration of a monomial M_v in formal ψ- and κ-classes of $\overline{M}_{g(v),n(v)}$. If j is the gluing map corresponding to the dual graph, the corresponding element of the generating set is given by

$$j_*\left(\prod_v M_v\right),$$

where here the formal monomials are evaluated at the usual ψ- and κ-classes.

By restriction we can also define tautological rings $R^*(M_{g,n})$, $R^*(M_{g,n}^{rt})$ and $R^*(M_{g,n}^{ct})$. From the above set of generators we see that the tautological ring of $R^*(M_g)$ is generated by κ-classes. Also we can define the tautological ring RH^* in cohomology analogously as in Chow.

2.3 Why?

It is very natural to ask why it is useful to study the tautological ring instead of the whole Chow or cohomology ring.

First, in low genus all classes turn out to be tautological: By results of Keel [29], in genus zero there are isomorphisms

$$R^*(\overline{M}_{0,n}) \cong A^*(\overline{M}_{0,n}) \cong H^*(\overline{M}_{0,n}) \cong RH^*(\overline{M}_{0,n}).$$

In genus one by results first announced by Getzler [16] and finally worked out by Petersen [42], the tautological ring in cohomology is still isomorphic to the even cohomology of $\overline{M}_{1,n}$. However already in this case for large enough n there exist odd cohomology classes which cannot be tautological by definition, and also the Chow ring will not be finitely generated.

The main reason for considering the tautological ring is that in most applications we use $\overline{M}_{g,n}$ because of its universal property and not just as an arbitrary space. Therefore when in an application a Chow or cohomology class is produced in $\overline{M}_{g,n}$ it is very likely to lie in the tautological ring. Knowledge about the tautological ring might help us with obtaining an explicit formula for this class or with comparing it to similar classes. In turn, as we will see in the next section, writing a class in terms of tautological generators is useful for intersection number computations.

Another reason why it is natural to study the tautological ring is Mumford's conjecture, which is now proven by Madsen-Weiss [33]. The statement of the conjecture is that the stable rational cohomology of M_g as $g \to \infty$ is isomorphic to

$$\mathbb{Q}[\kappa_1, \kappa_2, \dots]. \tag{2}$$

We do not want to make this precise here, but morally the conjecture implies that non-tautological classes are an unstable phenomenon, i.e. they don't come in canonical families for $g \to \infty$. Mumford's conjecture also implies that relations between tautological classes are unstable as $g \to \infty$.

3 Intersection numbers

Using intersection theory we can produce easier to understand objects from tautological classes: rational numbers

A class α in $A^{3g-3+n}(\overline{M}_{g,n})$ is just a finite collection of points with rational coefficients. Adding up the coefficients we obtain a rational number denoted by

$$\int_{\overline{M}_{g,n}} \alpha \in \mathbb{Q}$$

and called *intersection number* or *integral*. For this definition it was important that $\overline{M}_{g,n}$ is proper since we have actually taken the proper push-forward along the map from $\overline{M}_{g,n}$ to a point.

For the moduli spaces $M^{ct}_{g,n}$ and $M^{rt}_{g,n}$ $(g > 0)$ we cannot directly integrate. Still we can try to use the inclusions $M^{ct}_{g,n}, M^{rt}_{g,n} \subset \overline{M}_{g,n}$ and integration on $\overline{M}_{g,n}$. Here the classes λ_g and $\lambda_g\lambda_{g-1}$ are useful. The class λ_g has the property that it vanishes on the complement of $M^{ct}_{g,n}$ in $\overline{M}_{g,n}$, and $\lambda_g\lambda_{g-1}$ even vanishes on the complement of $M^{rt}_{g,n}$. So given $\alpha \in R^{2g-3+n}(M^{ct}_{g,n})$ or $\alpha \in R^{g-2+n}(M^{rt}_{g,n})$ we can choose any extension $\bar{\alpha}$ to $R^*(\overline{M}_{g,n})$ and still obtain well-defined integrals

$$\int_{\overline{M}_{g,n}} \bar{\alpha}\lambda_g \in \mathbb{Q}, \qquad \int_{\overline{M}_{g,n}} \bar{\alpha}\lambda_g\lambda_{g-1} \in \mathbb{Q},$$

respectively.

By the formula of Mumford λ_g and λ_{g-1} can be expressed in terms of other κ-, ψ- and boundary classes. In turn by using a generalization of (1) the κ-classes can be expressed in terms of ψ-classes. This essentially implies that all integrals of tautological classes can be expressed in terms of the basic integrals

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}, \tag{3}$$

where the k_i add up to $3g - 3 + n$.

Motivated from physics (2-dimensional quantum gravity) Witten proposed a conjecture [51], which makes it possible to compute all the intersection numbers (3). They are put into a generating series

$$F = \sum_{\substack{g,n\geq 0 \\ 2g-2+n>0}} \frac{1}{n!} \sum_{\substack{k_1,\ldots,k_n \\ \sum_{i=1}^n k_i = 3g-3+n}} \int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} t_{k_1} t_{k_2} \cdots t_{k_n} = \frac{t_0^3}{6} + \frac{t_1}{24} + \cdots$$

in infinitely many variables $t_0, t_1, \ldots$. Witten's conjecture then says that $\exp F$ is a τ-function for the KdV hierarchy. This means that F satisfies a system of infinitely many partial differential equations, which turn out to give enough information about F to recursively determine all coefficients of F and therefore the intersection numbers.

A first proof of Witten's conjecture has been given by Kontsevich [30] using a cellular decomposition of the moduli space obtained using Jenkins-Strebel differentials. From this he obtains a formula for certain specializations of $\exp F$ as a sum over ribbon graphs, which he interprets as certain $N \times N$-matrix integrals. He deduces that $\exp F$ must be a τ-function for the KdV hierarchy. The occurring matrix integral is an asymptotic expansion of a matrix generalization of the Airy function

$$\mathrm{Ai}(x) = \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) \mathrm{d}t. \tag{4}$$

By now there are many other proofs of Witten's conjecture. See for example [35, 28, 38].

4 Faber's conjectures

4.1 The conjectures

In [9] Carel Faber has proposed a remarkable series of conjectures concerning the structure of the tautological rings $R^*(M_g)$. Shortly afterwards similar conjectures about $R^*(M^{rt}_{g,n})$, $R^*(M^{ct}_{g,n})$ and $R^*(\overline{M}_{g,n})$ have been made [13, 10]. We will review here these conjectures together.

The first part of the conjectures says that each tautological ring is a Gorenstein $\mathbb{Q}$-algebra with socle in codimension

1. $N = g - 2 + n - \delta_{0,g}$ for $M^{rt}_{g,n}$,
2. $N = 2g - 3 + n$ for $M^{ct}_{g,n}$ and
3. $N = 3g - 3 + n$ for $\overline{M}_{g,n}$.

Here a graded $\mathbb{Q}$-algebra A^* is Gorenstein with socle in degree N if

1. for all $k > N$ the vector space A^k is trivial (*vanishing*),
2. there exists an isomorphism $A^N \cong \mathbb{Q}$ (*socle*) and
3. this isomorphism together with the intersection product defines a perfect pairing
$$A^k \times A^{N-k} \to A^N \cong \mathbb{Q}$$
for $0 \leq k \leq N$ (*perfect pairing*).

So the conjecture says that each tautological ring behaves like the rational cohomology ring of a compact manifold of dimension $2N$, in that it satisfies Poincaré duality.

The isomorphism between the socle and $\mathbb{Q}$ is given by the integrals on $\overline{M}_{g,n}$ discussed in the previous section.

The second conjecture of Faber says that $R^*(M_g)$, which as we have already seen is the ring generated by the κ-classes, is already generated by $\kappa_1, \ldots, \kappa_{\lfloor g/3 \rfloor}$ and that there are no relations between these classes in degree $\leq \lfloor \frac{g}{3} \rfloor$.

The third conjecture of Faber gives an explicit formula for the integrals of ψ-classes against $\lambda_g \lambda_{g-1}$:

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdot \cdots \cdot \psi_n^{k_n} \lambda_g \lambda_{g-1} = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)! \prod_{i=1}^n (2k_i - 1)!!} \int_{\overline{M}_{g,1}} \psi_1^{g-1} \lambda_g \lambda_{g-1}, \quad (5)$$

where $\sum k_i = g - 2 + n$. This conjecture is now known as the $\lambda_g \lambda_{g-1}$-conjecture. Getzler-Pandharipande have found in [17] a similar λ_g-conjecture using the degree zero Virasoro conjecture [8]:

$$\int_{\overline{M}_{g,n}} \psi_1^{k_1} \cdot \cdots \cdot \psi_n^{k_n} \lambda_g = \binom{2g-3+n}{k_1, \ldots, k_n} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g,$$

for $\sum k_i = 2g - 3 + n$. The integrals remaining on the right hand side of the λ_g- and $\lambda_g \lambda_{g-1}$-conjecture have been explicitly evaluated by Faber and Pandharipande [9, 11].

The Faber conjectures give an explicit description of the tautological ring as the *Gorenstein quotient*, i.e. the ring obtained from a formal polynomial ring in the generators by quotienting out the ideal of all formal homogeneous classes x which pair to zero with any class y of opposite degree. Since we have an explicit description of the set of generators and we can compute any integral of tautological classes, this gives a conjectural presentation of the tautological ring in terms of generators and relations.

Faber found his conjectures by developing an algorithm which finds infinitely many relations in the tautological ring of M_g. If one assumes that the relations obtained from the algorithm are random inside the space of all relations, sufficiently many of these relations should span the space of all relations with high probability. If the quotient of the polynomial ring of formal κ-classes by these relations becomes Gorenstein, one can stop because any relation pairs to zero with any class and therefore has to be zero in a Gorenstein ring.

4.2 Current status

By now most of Faber's conjectures have been proven. Let us walk through them in the opposite way that we have presented them.

In [17] it has already been shown that the λ_g- and $\lambda_g\lambda_{g-1}$-conjectures follow from the Virasoro conjecture for $\mathbb{P}^1$ and $\mathbb{P}^2$ respectively. Later Givental proved the Virasoro conjecture for projective spaces [18], thus completing the proof of the integral conjectures. The λ_g-conjecture had already been proven earlier in [12] via a localization computation for $\mathbb{P}^1$.

To prove the generation result it is necessary to find relations between κ-classes which express κ_i for $i > \lfloor \frac{g}{3} \rfloor$ in terms of polynomials of lower degree κ-classes. The first proof of the conjecture in cohomology is given by Morita in [36] using representation theory of the symplectic group. In Chow, the first proof of was found by Ionel in [26]. She obtains the necessary relations by an geometric argument using the locus of d-gonal curves in M_g.

The result that there are no relations between κ-classes in degree $\leq \lfloor \frac{g}{3} \rfloor$ is known as Harer stability since already before Faber's conjectures Harer proved a weaker result [23] in this direction. The statement necessary for Faber's conjecture has been proven by Boldsen in [3].

The most interesting part about Faber's conjecture is the Gorenstein property. The vanishing and socle part of the conjecture are now known in all cases. In [32] it is proven that $R^*(M_g)$ vanishes above degree $g-2$ and is at most one-dimensional in degree $g-2$. By showing that the integral of (5) is not zero Faber completed in [9] proof of the socle part for M_g. The work [22] shows the socle statement for $\overline{M}_{g,n}$ and in [14] uniform proofs for all vanishing and socle statements are given.

Given that the vanishing and socle part of the Gorenstein conjecture are known, proving the perfect pairing conjecture amounts to showing that the tautological ring is isomorphic to the Gorenstein quotient and not to a proper quotient of it.

As we have already noted, in genus zero and genus one the tautological ring of $\overline{M}_{g,n}$ coincides with the even cohomology ring and therefore Poincaré duality implies the Gorenstein conjecture. By results [48, 49] of Tavakol it is also known that $R^*(M^{ct}_{1,n})$ and $R^*(M^{rt}_{2,n})$ are Gorenstein.

By Faber's computations it is also known that $R^*(M_g)$ is Gorenstein for $g \leq 23$. However in genus 24, unless the "random" hundreds of relations computed by Faber span only a proper vector subspace of the space of relations, the ring $R^*(M_{24})$ has to coincide with the Gorenstein quotient except in degree 12, where $\dim R^{12}(M_{24}) = 37$ in contrast to $\dim R^{10}(M_{24}) = 36$. Other methods of producing relations as in [47] or [52] also did not find the "missing relation" in degree 12. Yin also studies the rings $R^*(M_{g,1})$ and finds that they are Gorenstein for $g \leq 19$, but did not find enough relations for $g = 20$.

With these results doubts about the validity of the Gorenstein conjecture began to arise and the following conjectures contradicting the Gorenstein conjecture were made: Pandharipande conjectured that in all relations for M_g can be expressed in terms of Faber-Zagier relations (see the next section). Furthermore, Yin conjectured that for $M_{g,1}$ all relations are of motivic nature. The conjecture of Pixton we discuss below is also in contradiction to the Gorenstein conjecture. For $\overline{M}_{g,n}$ and $M^{ct}_{g,n}$ all hope for the Gorenstein conjecture was completely lost with the genus 2 counterexamples of Petersen-Tommasi [43, 44].

Faber's conjectures do not make any statement about $M_{g,n}$ for $n \geq 2$. Here in fact, by results [5] of Buryak-Shadrin-Zvonkine, the tautological ring does not have a one-dimensional socle. Instead the classes $\psi_1^{g-1}, \ldots, \psi_n^{g-1}$ generate the n-dimensional vector space $R^{g-1}(M_{g,n})$. The vanishing $R^i(M_{g,n}) = 0$ for $i \geq g$ had already been shown earlier by Ionel in [27].

5 Relations

5.1 of Faber-Zagier

Faber's algorithm for relations in $R^*(M_g)$ produced in all examples a unique relation in degree d for $g = 3d+1$. In 2000, after many computer calculations and with a lot of guesswork, he together with Zagier conjectured a general formula for this relation as well as an explicit set of polynomials in κ-classes that all other relations found in the calculations are linear combinations of. The hypergeometric series

$$A(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} t^i,$$

$$B(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} t^i$$

play an important role in their formulation. The conjectural relations are parametrized by a non-negative integer r and a partition σ with no part equal to 2 modulo 3, such that $g-1+|\sigma|<3r$ and $g \equiv r+|\sigma|+1 \pmod 2$. It takes some definitions to write down the relations:

Let

$$\mathbf{p} = \{p_1, p_3, p_4, p_6, p_7, \dots\}$$

be a collection of variables indexed by the positive integers not congruent to 2 modulo 3 and let $\Psi(t,\mathbf{p})$ be the formal power series

$$A(t)\sum_{i=0}^{\infty} t^i p_{3i} + B(t)\sum_{i=0}^{\infty} t^i p_{3i+1},$$

where $p_0 := 1$. Define rational numbers $C_d(\sigma)$, for σ any partition with parts not congruent to 2 modulo 3, by the formula

$$\log(\Psi(t,\mathbf{p})) = \sum_{\sigma}\sum_{d=0}^{\infty} C_d(\sigma) t^d \mathbf{p}^\sigma,$$

where $\mathbf{p}^\sigma$ denotes $p_1^{a_1} p_3^{a_3} p_4^{a_4} \cdots$ if σ is the partition $[1^{a_1} 3^{a_3} 4^{a_4} \cdots]$. Set

$$\gamma = \sum_{\sigma}\sum_{d=0}^{\infty} C_d(\sigma)\kappa_d t^d \mathbf{p}^\sigma.$$

Then the $t^d\mathbf{p}^\sigma$-coefficient of $\exp(-\gamma)$ is the Faber-Zagier relation for (d,σ). The unique degree d relation in $R^*(M_{3d+1})$ can be obtained by taking σ to be the empty partition.

Faber-Zagier had already proved in 2002 that their conjectural relations are at least relations on the level of the Gorenstein quotient, i.e. they pair to zero with all classes of complementary codimension. However to my knowledge [46] is the only place where to find a written down proof of this fact.

In [39] the first proof of the fact that these polynomials in κ-classes are actual tautological relations was presented. The proof uses the geometry of the moduli space of stable quotients (see Section 6) and analysis of the generating series

$$\Phi(z,q) = \sum_{d=0}^{\infty}\prod_{i=1}^{d}\frac{1}{1-iz}\,\frac{(-1)^d}{d!}\frac{q^d}{z^d}, \tag{6}$$

which continues the analysis of the same series in [26].

The series A and B are fundamental to the tautological ring of M_g. They very directly give the coefficients of the unique lowest degree relations. After first being found by experimentally by Faber-Zagier, the series were derived in different ways in [36, 26, 40]. As I noticed from [40], the series actually appear directly in the asymptotic expansion of the Airy function and its first derivative:

$$\mathrm{Ai}(x) \asymp_{x\to\infty} \frac{\sqrt{\pi}}{2} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} A\left(-\frac{x^{-\frac{3}{2}}}{576}\right),$$

$$\mathrm{Ai}'(x) \asymp_{x\to\infty} \frac{\sqrt{\pi}}{2} x^{\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} B\left(-\frac{x^{-\frac{3}{2}}}{576}\right).$$

It is interesting to see the Airy function appearing naturally in the context of the intersection numbers and the relations. See [4] for a survey of these and similar appearances.

5.2 of Pixton

In 2012 Pixton was able to find relations [45] in the Gorenstein quotient of $\overline{M}_{g,n}$, which, in the case $n = 0$ and when restricted to the locus of smooth curves, exactly recover the relations of Faber-Zagier. We do not write them explicitly here, but just note that they have a contribution from each boundary stratum which in turn is a product of component, marking and node contributions. All the contributions are determined in terms of the series A and B. Pixton's relations behave well under push-forward and pull-back via the tautological maps and all known explicit relations in the tautological ring can be written in terms of his relations. He conjectured that his relations are relations on the level of the tautological ring and that they are all of the relations.

The first proof of the fact that they are relations in the tautological ring in cohomology has been given by Pandharipande-Pixton-Zvonkine in [40]. We will give the idea of proof in Section 7. Paper A contains the first proof in Chow.

See [46] for some computations showing the discrepancy between the tautological ring according to Pixton's conjecture and the Gorenstein quotient. For example the first discrepancy for $R^*(M_{g,1})$ is for $g = 20$ — exactly the same place as where Yin was missing a relation. It is still very unclear how to approach Pixton's conjecture that his relations are all the tautological relations.

6 Stable maps and quotients

Gromov-Witten theory tries to study a space X (say a symplectic manifold or a smooth variety) by studying in some sense the number of isomorphism classes of maps from curves C to X satisfying some boundary conditions. To X the deformation invariant *Gromov-Witten invariants* are assigned. One of the first applications of Gromov-Witten theory was the computation of the number of rational curves of degree d through $3d-1$ points of $\mathbb{P}^2$ in generic position.

The moduli space underlying Gromov-Witten theory is $\overline{M}_{g,n}(X,\beta)$, the moduli space of *stable maps* $f : C \to X$ from nodal curves C of genus g with n markings to X such that $f([C])$ has class $\beta \in H_2(X;\mathbb{Z})$. So the moduli space parametrizes marked curves together with a map. As for curves, "stable" means that each object (C,f) should only have finitely many automorphisms. One feature of stable maps is that there can be contracted components, i.e. some of the components of C can be mapped to a point.

The moduli space $\overline{M}_{g,n}(X,\beta)$ is still a Deligne-Mumford stack and there is a proper forgetful map ν to $\overline{M}_{g,n}$. However $\overline{M}_{g,n}(X,\beta)$ has components of varying dimensions and is therefore far from being smooth. In particular there are in general components of the moduli of higher dimension than the "generic" component of stable maps contracting none of the components of the domain curve. The dimension of the "generic" component

$$\mathrm{vdim} = (\dim(X)-3)(1-g) + \int_\beta c_1(T_X).$$

is known as the expected or *virtual dimension.* In [31] a program was proposed to give an algebraic definition of Gromov-Witten invariants by constructing a cycle

$$[\overline{M}_{g,n}(X,\beta)]^{vir} \in A_{\mathrm{vdim}}(\overline{M}_{g,n}(X,\beta))$$

satisfying many axioms. Such a cycle is called a *virtual fundamental class.* In [2] a suitable cycle was constructed.

Gromov-Witten invariants are defined via integration against the virtual class. For the *primary invariants*, the only classes used in the integration are cycle classes of X pulled-back via the *evaluation maps* ev_i ($i \in \{1,\ldots,n\}$), i.e. the map sending a stable map to the image of the ith marking to its image in X. For *descendent Gromov-Witten invariants*, in addition cotangent line classes at the markings of the source curve are allowed.

In this thesis we will use Gromov-Witten theory to obtain information about the moduli space of curves. Instead of directly computing intersection numbers, which is equivalent to computing the push-forward with respect to the forgetful map from $\overline{M}_{g,n}(X,\beta)$ to a point, we can factor this process into first performing the push-forward via ν and then computing intersection numbers on $\overline{M}_{g,n}$. In this thesis a main interest will lie in computing the classes in $\overline{M}_{g,n}$ obtained from the push-forward via ν of classes of $\overline{M}_{g,n}(X,\beta)$ capped against the virtual fundamental class. Because of its simplicity, the main target X we consider will be $\mathbb{P}^1$. As in general for toric targets, the method of virtual localization [21] makes it possible to compute all these push-forwards explicitly.

To add some more detail, we will consider a non-trivial $\mathbb{C}^*$-action on $\mathbb{P}^1$, which induces a $\mathbb{C}^*$-action on $\overline{M}_{g,n}(\mathbb{P}^1,d)$. Here, d is an abbreviation for the dth multiple of the fundamental class. By [21] the virtual fundamental class splits into a sum of contributions from each fixed locus of $\overline{M}_{g,n}(\mathbb{P}^1,d)$. Each fixed locus is essentially a product of moduli spaces of curves and the contributions can be expressed in terms of tautological classes.

In [39] and Paper A, the closely related moduli space of *stable quotients* to $\mathbb{P}^1$ is used. The idea is to replace maps $f : C \to \mathbb{P}^1$ by the pull-back of the universal sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathcal{O}_{\mathbb{P}^1}^2 \to \mathcal{O}_{\mathbb{P}^1}(1) \to 0$$

via f. The moduli problem considers marked nodal curves C together with a quotient sequence

$$0 \to S \to \mathcal{O}_C^2 \to Q \to 0$$

such that S is a locally free sheaf of rank one and Q is a coherent sheaf locally free at the nodes and markings of C. In addition a stability condition is imposed. So whereas for a stable map the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ is a vector bundle, Q is allowed to have torsion.

A suitable moduli space $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ is constructed in [34] using Grothendieck's Quot-scheme. Here d stands for the degree of the quotient Q. In addition, in analogy with the moduli space of stable maps, a forgetful map to the moduli space of curves, evaluation maps and a virtual fundamental class are constructed. Furthermore, $\overline{Q}_{g,n}(\mathbb{P}^1,d)$ is birational to $\overline{M}_{g,n}(\mathbb{P}^1,d)$, there is a comparison map

$$c : \overline{M}_{g,n}(\mathbb{P}^1,d) \to \overline{Q}_{g,n}(\mathbb{P}^1,d)$$

and all analogously defined invariants or push-forwards to the moduli space of curves coincide. The main reason why $\overline{Q}_{g,n}(\mathbb{P}^1, d)$ is used in [39] is because in a localization computation less fixed loci will need to be considered compared to $\overline{M}_{g,n}(\mathbb{P}^1, d)$

7 Cohomological field theories

The classes one obtains from the moduli space of stable maps via push-forward along the forgetful map to the moduli space of curves satisfy many properties, which follow from the properties of the virtual fundamental class. The notion of a *cohomological field theory* [31] captures most of these.

Given a finite dimensional vector space V, a non-singular bilinear form η and a vector $\mathbf{1}$, a cohomological field theory (CohFT) consists of a symmetric multilinear form

$$\Omega_{g,n} \in H^*(\overline{M}_{g,n}) \otimes (V^*)^{\otimes n}$$

for every g and n, satisfying some axioms, which basically say that the multilinear forms behave nicely with respect to pull-back via the gluing and forgetful maps.

In the example of the Gromov-Witten theory of a smooth variety X (assuming for simplicity that X has no odd cohomology) the vector space V is the cohomology ring of X, the bilinear form η is the Poincaré pairing, $\mathbf{1}$ is the identity and for $\alpha_1, \ldots, \alpha_n \in V$ we have

$$\Omega_{g,n}(\alpha_1, \ldots, \alpha_n) = \sum_\beta \nu_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i) \cap [\overline{M}_{g,n}(X, \beta)]^{vir} \right),$$

where ν is the projection to the moduli space of curves. For this definition we need to assume that the sum over β converges. In any case, generalizing the definition of a CohFT slightly, we can also add some formal variables q^β to record β into the sum, and view $\Omega_{g,n}$ as a CohFT over the *Novikov ring*, i.e. the formal power series ring generated by variables q^β satisfying $q^{\beta_1} q^{\beta_2} = q^{\beta_1+\beta_2}$.

There are also CohFTs not coming from Gromov-Witten theory. For example in [40] a beautiful two dimensional example, Witten's 3-spin class $W_{g,n}$ is used. For the purposes of the relations a very important fact is that it is of pure cohomological degree

$$\deg(W_{g,n}(e_{a_1}, \ldots, e_{a_n})) = \frac{g - 1 + \sum_{i=1}^n a_i}{3}, \tag{7}$$

where each of $a_1, \dots, a_n$ is 0 or 1 and $\{e_0, e_1\}$ is a distinguished basis of the underlying vector space. For $r \geq 4$ there is also an $(r-1)$-dimensional CohFT defined using Witten's r-spin class.

Integrating the classes of a CohFT can be used to give V, thought of as a manifold, the structure of a Frobenius manifold as introduced in [7]. In particular every tangent space has the structure of a commutative Frobenius algebra. The points where this algebra structure does not have any non-zero nilpotent elements are called *semisimple*.

In the case that the origin in V is semisimple a powerful conjectural reconstruction result of Givental [18] applies. It gives a formula to compute $\Omega_{g,n}$ in terms of genus 0 data which is computable in most cases. Givental's conjecture is now proven in cohomology by Teleman in [50]. As the proof uses Mumford's conjecture, it cannot be directly generalized to Chow. Still in some examples such as the Gromov-Witten theory of toric varieties it is known to also hold in Chow [19].

The 3-spin CohFT used in [40] is not semisimple at the origin, but at least semisimple at a generic point of the Frobenius manifold. By a shifting procedure the CohFT can be moved to such a point and Teleman's reconstruction result can be applied there. Pixton's relations are found by the observation that while shifted Witten's class is supported in cohomological degree at most (7) the reconstruction gives terms of higher degree. These terms of higher degree must cancel, but this cancellation is non-trivial and very directly implies the relations of Pixton.

8 Summary of the papers

8.1 Paper A

Paper A gives a proof of the fact that Pixton's relations are relations in the tautological ring in Chow.

The proof takes the stable quotient relations of [39], which, as the authors point out, are restrictions of relations of $R^*(\overline{M}_g)$. For concreteness, on the way of generalizing to $\overline{M}_{g,n}$ first the case of C_g^n, the n-fold tensor power of the universal curve over M_g, is considered. In this case the proof is, up to minor simplifications, the same as in [39].

When trying to go to the compactification $\overline{M}_{g,n}$ things get much more complicated. Whereas in the stable quotient localization computation for C_g^n

only two fixed loci play a role, for $\overline{M}_{g,n}$, as the degree of the stable quotient grows, more and more fixed loci need to be considered.

The fixed loci can be sorted according to dual graphs but at each edge and leg of the dual graph we still need to sum over localization contributions of chains of rational components. As pointed out by my advisor these sums should be computable by the methods of Givental [19]. Using the comparison between stable quotient and stable map invariants the series can indeed be computed. At the time it was astonishing that the series $\Phi(z, q)$ of (6) reappears here.

To get enough relations, in all cases it is necessary to not only consider the virtual fundamental class but also its intersection with some natural Chow classes on the moduli space of stable quotients. In the cases of C_g^n and $\overline{M}_{g,n}$ different classes are considered. In the first case the first Chern class of the universal sheaf and in the second case, like in Gromov-Witten theory, pull-backs of classes of $\mathbb{P}^1$ via the evaluation maps are taken. In order to understand why in the end the same relations are obtained, Hassett's moduli space [25] of curves with weighted markings is considered to interpolate between these cases.

8.2 Paper B

The work on Paper B was prompted by discussions with Y.P. Lee at the conference *Cohomology of the moduli space of curves* organized by the *Forschungsinstitut für Mathematik (FIM)* at ETH Zürich. Via the comparison between stable maps and stable quotients, the localization calculation of Paper A is essentially equivalent to Givental's proof [19] of the reconstruction theorem for the Gromov-Witten theory of $\mathbb{P}^1$.

As remarked in [40], the way that the relations are found from the 3-spin theory can also be interpreted in a different way: While the Givental-Teleman reconstruction cannot be directly applied at the non-semisimple origin, we can shift to a semisimple point and look at what happens when we try to approach the non-semisimple point. On the one hand the individual terms of the reconstruction diverge, but on the other hand we know that the result has to be Witten's class. It follows that there has to be cancellation between the terms.

As suggested by D. Zvonkine, this method can be used for any CohFT which is semisimple at a generic point of the Frobenius manifold. The terms in the reconstruction have poles at the non-semisimple locus and the necessary cancellation in order for a non-semisimple limit to exist imply tautological relations. This approach gives a rich procedure of finding tautological relations and while the relations have similar origin, they are much harder to make explicit in higher dimensions and it is not directly clear how they are related to Pixton's relations.

Paper B gives a first comparison between these relations from different CohFTs. It is shown that the relations from Witten's 3-spin class (i.e. Pixton's relations) are equivalent to the relations from the equivariant Gromov-Witten theory of $\mathbb{P}^1$. Furthermore, in higher dimensions the relations from Witten's r-spin class are contained in the set of relations of the equivariant Gromov-Witten theory of $\mathbb{P}^r$.

In higher dimensions, it is not clear that relations from degree considerations as described in Section 7 are the same as relations from pole cancellation. In Paper B, it is at least shown that the relations from the degree considerations are included in the set of relations from pole cancellation.

In [41] (in preparation) the relations from degree considerations for Witten's r-spin class are explicitly computed for specific shifts on the Frobenius manifold. The relations from Witten's 4-spin class are especially interesting since they are more tractable than Pixton's relations and by [46] the fact that $\dim RH^{2(g-2)}(M_g) = 1$ can already be derived only from these relations. The results of Paper B show that these relations hold also in Chow.

8.3 Paper C

Paper C finally proves a strong comparison result for the relations from pole cancellation: The relations from any generically semisimple but not everywhere semisimple CohFT are equivalent to Pixton's relations.

In particular, all higher r-spin relations can be expressed in terms of Pixton's relation, as well as many relations obtained from virtual localization. For example in future work [6] connection to the relations of Randal-Williams [47] will be made.

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Part II
Scientific Papers

Paper A

Tautological relations in moduli spaces of weighted pointed curves

arXiv:1306.6580v2

TAUTOLOGICAL RELATIONS IN MODULI SPACES OF WEIGHTED POINTED CURVES

FELIX JANDA

Abstract. Pandharipande-Pixton have used the geometry of the moduli space of stable quotients to produce relations between tautological Chow classes on the moduli space M_g of smooth genus g curves. We study a natural extension of their methods to the boundary and more generally to Hassett's moduli spaces $\overline{M}_{g,\mathbf{w}}$ of stable nodal curves with weighted marked points.

Algebraic manipulation of these relations brings them into a Faber-Zagier type form. We show that they give Pixton's generalized FZ relations when all weights are one. As a special case, we give a formulation of FZ relations for the n-fold product of the universal curve over M_g.

1. Introduction

1.1. Moduli spaces of curves with weighted markings.

1.1.1. *Definition.* As a GIT variation of the Deligne-Mumford moduli space of stable marked curves, for any n-tuple $\mathbf{w} = (w_1, \cdots, w_n)$ with $w_i \in \mathbb{Q}\cap]0, 1]$ Hassett [10] has defined a moduli space $\overline{M}_{g,\mathbf{w}}$, parametrizing nodal semi-stable curves C of arithmetic genus g with n numbered marked points $(p_1, \dots, p_n)$ in the smooth locus of C satisfying two stability conditions:

(1) The points in a subset $S \subseteq \{1, \dots, n\}$ are allowed to come together if and only if $\sum_{i\in S} w_i \leq 1$.
(2) $\omega_C\left(\sum_{i=1}^n w_i p_i\right)$ is ample.

The second condition implies that the total weight plus the number of nodes of every genus 0 component of C must be strictly greater than 2.

The main cases we have in mind are the usual moduli space $\overline{M}_{g,n}$ of marked curves, which occurs when all the weights are equal to 1, the case when $\sum_{i=1}^n w_i \leq 1$, which is a desingularization of the n-fold product of the universal curve over $\overline{M}_g$, and the case when $g = 0$, $w_1 = 1$, $w_2 = 1$ and $\sum_{i=3}^n w_i \leq 1$, which gives the Losev-Manin spaces [13]. Moduli spaces mixed pointed curves also naturally appear when studying moduli spaces of stable quotients [14].

We will use various abbreviations for the weight data, like $(\mathbf{w}, 1^m)$ for the data with first entries given by $\mathbf{w}$ and further m entries of 1.

1.1.2. *Tautological classes.* Using the universal curve $\pi : \overline{C}_{g,\mathbf{w}} \to \overline{M}_{g,\mathbf{w}}$, the n sections $s_i : \overline{M}_{g,\mathbf{w}} \to \overline{C}_{g,\mathbf{w}}$ corresponding to the markings and the relative dualizing sheaf ω_π we can define ψ- and κ-classes:

$$\psi_i = c_1(s_i^* \omega_\pi)$$
$$\kappa_i = \pi_*(c_1(\omega_\pi)^{i+1})$$

Notice that in the case of $\overline{M}_{g,n}$ the definition of κ-classes is different from the usual definition as in [2].[1]

Each subset $S \subseteq \{1, \dots, n\}$ defines a diagonal class D_S as the class of the locus where all the points of S coincide. By Condition (1) the class D_S is zero if and only if $\sum_{i \in S} w_i > 1$.

As $\overline{M}_{g,n}$ the moduli space $\overline{M}_{g,\mathbf{w}}$ is stratified according to the topological type of the curve and each stratum, indexed by a dual graph Γ (see [8, Appendix A] for a description of dual graphs of strata of $\overline{M}_{g,n}$), is the image of a clutching map

$$\xi_\Gamma : \prod_i \overline{M}_{g_i,\mathbf{w}_i} \to \overline{M}_{g,\mathbf{w}}.$$

Here points which are glued together have weight 1. The map ξ_Γ is finite of degree $|\operatorname{Aut}(\Gamma)|$ [2].

1.2. **Formulation of the relations.** To state the Faber-Zagier-type relations on $\overline{M}_{g,\mathbf{w}}$ we need to introduce several formal power series.

The hypergeometric series A and B already appeared in the original FZ relations. They are defined by

$$A(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{t}{72}\right)^i = 1 + O(t^1), \quad B(t) = \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{6i+1}{6i-1} \left(\frac{t}{72}\right)^i.$$

We will actually not directly use B in the definition of the relations but A and a family C_i of series strongly related to A and B which were already used in [19] for the proof of the equivalence between stable quotient and FZ relations on M_g. They are defined recursively by

$$C_1 = C = \frac{B}{A}, \quad C_{i+1} = \left(12t^2 \frac{\mathrm{d}}{\mathrm{d}t} - 4it\right) C_i.$$

Notice that C_i is a multiple of t^{i-1}.

As in [19] these series appear in the study of the two variable functions

$$\Phi(t,x) = \sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1-it} \frac{(-1)^d}{d!} \frac{x^d}{t^d}$$
$$\gamma = \sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i-1)} t^{2i-1} + \log(\Phi),$$

where the Bernoulli numbers B_k are defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

[1]The κ-classes we use here appear naturally when pushing forward powers of ψ-classes along maps forgetting points of small weight whereas the usual κ-classes are convenient when studying push-forwards of powers of ψ-classes along maps forgetting points of weight equal to 1. The fact that we will mainly consider the first kind of push-forwards explains our choice of κ-classes.

[2]To determine the number $|\operatorname{Aut}(\Gamma)|$ of automorphisms of Γ the graph Γ should be regarded as a collection of distinct half-edges of which some are glued together. For example when $n = 0$ and Γ consists of exactly one vertex and one edge, there is exactly one non-trivial automorphism, which interchanges the two half-edges; accordingly the map $\xi_\Gamma : \overline{M}_{g-1,2} \to \overline{M}_g$ is a double cover.

These two-variable functions appear both in the localization formula for stable quotients (see Section 3 and [19]) and the S-matrix in the equivariant genus 0 Gromov-Witten theory of $\mathbb{P}^1$ (see Section 5).

Various linear bracket operators are used to insert Chow classes into the power series in t. We define $\{\}_\kappa$, $\{\}_\psi$ and $\{\}_{D_S}$ for $S \subseteq \{1,\ldots,n\}$ by

$$\begin{aligned} \{t^k\}_\kappa =& \kappa_k t^k \\ \{t^k\}_\psi =& \psi^k t^k \\ \{t^k\}_{D_S} =& \begin{cases} \psi_i^k t^k, & \text{if } S=\{i\}, \\ (-1)^{|S|-1} D_S \psi_i^{k-|S|+1} t^k, & \text{else, where } i \in S, \end{cases} \end{aligned}$$

and linearity. It will moreover be useful to define brackets modified by a sign $\zeta \in \{\pm 1\}$, denoted by $\{\}_\kappa^\zeta$, $\{\}_\psi^\zeta$ and $\{\}_{D_S}^\zeta$ respectively, by composing the usual bracket operator with the ring map induced by $t \mapsto \zeta t$. For a power series F in t we will use the notation $[F]_{t^i}$ for the t^i coefficient of F.

Proposition 1. *For any codimension r and the choice of a subset $S \subseteq \{1,\ldots,n\}$ such that $3r \geq g+1+|S|$ the class*

$$\left[\sum_{\substack{\Gamma \\ \zeta:\Gamma\to\{\pm1\}}} \frac{1}{|\operatorname{Aut}(\Gamma)|}\xi_{\Gamma *}\Big(\prod_{v \text{ vertex}} \exp(-\{\log(A)\}_{\kappa^{(v)}}^{\zeta(v)}) \sum_{P \vdash S_v} \prod_{i \in P} \{C_{|i|}\}_{D_i}^{\zeta(v)} \prod_{e \text{ edge}} \Delta_e\Big)\right]_{t^{r-|E|}}$$

in $A^r(\overline{M}_{g,\mathbf{w}})$ is zero, where the sum is taken over all dual graphs Γ of $\overline{M}_{g,\mathbf{w}}$ with vertices colored by ζ with $+1$ or -1, the class $\kappa_i^{(v)}$ is the i-th κ-class in the factor corresponding to v and S_v is S restricted to the markings at v. The edge term Δ_e depends only on the ψ-classes ψ_1, ψ_2 and colors $\zeta_i = \zeta(v_i) \in \{\pm 1\}$ at the vertices v_1, v_2 joined by e and is defined by

$$2t(\psi_1+\psi_2)\Delta_e = (\zeta_1+\zeta_2)\{A^{-1}\}_{\psi_1}^{\zeta_1}\{A^{-1}\}_{\psi_2}^{\zeta_2} + \zeta_1\{C\}_{\psi_1}^{\zeta_1} + \zeta_2\{C\}_{\psi_2}^{\zeta_2}.$$

To see that the series Δ_e is well-defined one can use the identity $A(t)B(-t) + A(-t)B(t) + 2 = 0$ [21]. The proof of Proposition 1 gives an alternative more geometric proof.

In the case of $\overline{M}_{g,n}$ the relations of Proposition 1 are a reformulation of the part of Pixton's generalized FZ relations [21] with empty partition σ and coefficients a_i only valued in $\{0,1\}$. The set S corresponds to the set of all i such that $a_i = 1$.

To obtain a set of relations analogous to Pixton's relations we need to take the closure of the relations of Proposition 1 under multiplication with ψ- and κ-classes and push-forward under maps forgetting marked points of weight 1. See for this also the discussion in Section 6.4 and [20, Section 3.5].

In total this gives a proof of [21, Conjecture 1] in Chow. We therefore have verified Pixton's remark [21] that it should be possible to adapt the stable quotients method to prove that his generalized relations hold. In cohomology [21, Conjecture 1] has already been established by a completely different method in [20].

1.3. **Plan of Paper.** Section 2 introduces stable quotient moduli spaces of $\mathbb{P}^1$ with weighted marked points, slightly generalizing the usual moduli spaces of stable quotients. We define them, sketch their existence and review structures on them. In Section 3 we review the virtual localization formulas both for stable quotients and stable maps to $\mathbb{P}^1$. Section 4 contains a proof of Proposition 1 restricted to powers of the universal curve over M_g. The proof in this case is much simpler than the general case but many parts of it can be referred to later on. Section 5 provides calculations of localization sums using Givental's method necessary in the proof of the general relations. Finally Section 6 contains the proof of Proposition 1.

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2. Moduli spaces of stable quotients

2.1. **Introduction.** The proof of Proposition 1 is based on the geometry of stable quotients. These moduli spaces give an alternative compactification of the space of maps from curves to Grassmannians and were first introduced in [14]. We will need a combination of these spaces with Hassett's spaces of weighted stable curves. These spaces are different from the ϵ-stable quotient spaces introduced by Toda [22], where the stability conditions on quotient sheaf instead of the points varies. Similar spaces $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ in Gromov-Witten theory have been studied in [1], [3] and [16].

The moduli space $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ parametrises nodal curves C of arithmetic genus g with n markings p_i weighted by $\mathbf{w}$ together with a quotient sequence

$$0 \to S \to \mathcal{O}_C \otimes \mathbb{C}^2 \to Q \to 0$$

satisfying several conditions:

- The underlying curve with weighted marked points satisfies all the properties of being stable but possibly the ampleness condition (2).
- S is locally free of rank 1.
- Q has degree d.
- The torsion of Q is outside the nodes and the markings of weight 1.
- $\omega_C\left(\sum_{i=1}^n w_i p_i\right) \otimes S^{\otimes(-\varepsilon)}$ is ample for any $0 < \varepsilon \in \mathbb{Q}$.

Isomorphisms of stable quotients are defined by isomorphisms of weighted stable curves such that the kernels of the quotient maps are related via pull-back by the isomorphism. In Section 2.3 we give a precise definition for families and sketch a proof that $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ is a proper Deligne-Mumford stack.

2.2. **Structures.** There is a universal curve $\overline{C}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ over $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ with a universal quotient sequence

$$0 \to \mathcal{S} \to \mathcal{O}_{\overline{C}_{g,\mathbf{w}}(\mathbb{P}^1,d)} \otimes \mathbb{C}^2 \to \mathcal{Q} \to 0.$$

Moreover as in Gromov-Witten theory for each marking i of weight 1 there is an evaluation map $\mathrm{ev}_i : \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d) \to \mathbb{P}^1$, which is defined by noticing that

$$0 \to \mathcal{S} \to \mathcal{O}_C \otimes \mathbb{C}^2 \to \mathcal{Q} \to 0$$

restricted to a marking p of weight 1 gives a point in $\mathbb{P}^1$ since tensoring the above sequence with the residue field k_p at p gives a k_p valued point in the Grassmannian $\mathbb{G}(1,2) = \mathbb{P}^1$.

There is also a forgetful map $\nu : \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d) \to \overline{M}_{g,\mathbf{w}}$ forgetting the data of the quotient sequence and stabilizing unstable components.

Finally, there is a comparison map $c : \overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d) \to \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ from the moduli space of stable maps of degree d to $\mathbb{P}^1$. It contracts all components that would become unstable [3] and introduces torsion at the point the component is contracted to according to the degree of the map restricted to that component.

The open substack $Q_{g,\mathbf{w}}(\mathbb{P}^1, d) \subset \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ is defined to be the preimage of the moduli space of smooth curves $M_{g,\mathbf{w}}$ under the forgetful map ν.

2.3. **Construction.** We want to reduce the existence of moduli spaces of stable quotients with weighted marked points to that of the usual moduli spaces of stable quotients. For this we use ideas from [1] and [3]. Because it is not the main topic of this article we will try to be as brief as possible.

We will allow in this section the weight data also to include 0. So with $\mathbf{w}$ we will denote an n-tuple $(w_1, \dots, w_n)$ with $0 \leq w_i \leq 1$ for all $i \in \{1, \dots, n\}$ here. We also consider more generally stable quotients to $\mathbb{P}^m$.

Definition. An object $(C, s_1, \dots, s_n, \mathcal{O}_C^{m+1} \to \mathcal{Q})$ in the stack $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ over a scheme S is a proper, flat morphism $\pi : C \to S$ together with n sections s_i and a quotient sequence of quasi-coherent sheaves on C flat over S

$$0 \to \mathcal{S} \to \mathcal{O}_C^{m+1} \to \mathcal{Q} \to 0$$

such that

(1) The fibers of π over geometric points are nodal connected curves of arithmetic genus g.
(2) For any $S \subset \{1, \dots, n\}$ such that the intersection of s_i for all $i \in S$ is nonempty we must have $\sum_{i \in S} w_i \leq 1$ and if in addition the intersection touches the singular locus of π we must have $\sum_{i \in S} w_i = 0$.
(3) $\mathcal{S}$ is locally free of rank 1.
(4) $\mathcal{Q}$ is locally free outside the singular locus of π and of degree d.
(5) $\omega_\pi(\sum_{i=1}^n w_i s_i) \otimes \mathcal{S}^{\otimes(-\varepsilon)}$ is π relatively ample for any $\varepsilon > 0$.

Two families $(C, s_1, \dots, s_n, \mathcal{O}_C^{m+1} \to \mathcal{Q})$, $(C', s'_1, \dots, s'_n, \mathcal{O}_{C'}^{m+1} \to \mathcal{Q}')$ of stable quotients over S are isomorphic if there exists an isomorphism $\phi : C \to C'$ over S mapping s_i to s'_i for all $i \in \{1, \dots, n\}$ and such that S and $\phi^*(S')$ coincide when viewed as subsheaves of $\mathcal{O}_{C'}^{m+1}$.

Notice that there is no condition on the sections of weight 0. Therefore the space $\overline{Q}_{g,(\mathbf{w},0^f)}(\mathbb{P}^m, d)$ is isomorphic to the f-fold power of the universal curve of $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ over $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$.

In order to prove that $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ is a Deligne-Mumford stack it is enough to show that it is locally isomorphic to a product of universal curves over $\overline{Q}_{g,1^f}(\mathbb{P}^m, d)$ since this is a Deligne-Mumford stack as shown in [14] (by realizing it as a stack quotient of a locally closed subscheme of a relative Quot scheme).

[3]The only components which are stable in the Gromov-Witten but not in the stable quotients theory are non-contracted components of genus 0 with exactly one node and no marking.

Lemma 1. *For each point* $(C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to \mathcal{Q})$ *in* $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ *there exists an open neighborhood which is isomorphic to an open substack of* $\overline{Q}_{g,\mathbf{w}'}(\mathbb{P}^m, d)$ *where* $\mathbf{w}'$ *is* $\{0,1\}$ *valued.*

Proof. The argument is the same as in [1, Corollary 1.18]. □

Separatedness follows from the following lemma, which is analogous to [3, Proposition 1.3.4].

Lemma 2. *The diagonal* $\Delta : \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d) \to \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d) \times \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ *is representable, finite and separated.*

Proof. We proceed as in [3, Proposition 1.3.4].

Let $(C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to \mathcal{Q})$ and $(C', s'_1, \ldots, s'_n, \mathcal{O}_{C'}^{m+1} \to \mathcal{Q}')$ be two stable quotients over a base scheme S. We need to show that the category

$$\mathrm{Isom}((C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to \mathcal{Q}), (C', s'_1, \ldots, s'_n, \mathcal{O}_{C'}^{m+1} \to \mathcal{Q}'))$$

is represented by scheme finite and separated over S.

The images T and T' of the maps $(\mathcal{O}_C^{m+1})^* \to S^*$ and $(\mathcal{O}_{C'}^{m+1})^* \to (S')^*$ are given by $T = S^*(-D)$, $T' = (S')^*(-D')$ for effective divisors D, D' of some degree $d' \le d$ on C respectively C'. From this we can construct d' additional sections $s_{n+1}, \ldots, s_{n+d'}$ for C and $s'_{n+1}, \ldots, s'_{n+d'}$ for C'. As in [4, Proposition 1.3.4] at least étale locally $d - d'$ [4] further sections $s_{n+d'}, \ldots, s_{n+d}$ for C and $s'_{n+d'}, \ldots, s'_{n+d}$ for C' can be constructed by choosing suitable hyperplanes H_i in $\mathbb{C}^{m+1}$ and marking sections at which the quotient is locally free and the quotient sequence coincides with the sequence corresponding to the inclusion of H_i in $\mathbb{C}^{m+1}$.

By the definition of stable quotients the resulting marked curves $(C, s_1, \ldots, s_{n+d})$ and $(C', s'_1, \ldots, s'_{n+d})$ are $(\mathbf{w}, \varepsilon^d)$-stable. This gives a closed embedding

$$\mathrm{Isom}((C, s_1, \ldots, s_n, \mathcal{O}_C^{m+1} \to \mathcal{Q}), (C', s'_1, \ldots, s'_n, \mathcal{O}_{C'}^{m+1} \to \mathcal{Q}')) \hookrightarrow$$
$$\bigsqcup_{\sigma \in S_{d'}} \mathrm{Isom}((C, s_1, \ldots, s_{n+d}), (C', s'_1, \ldots, s'_n, s'_{n+\sigma(1)}, \ldots, s'_{n+\sigma(d')}, s'_{n+d'+1}, \ldots, s'_{n+d})).$$

Since by [10] the right hand side is a scheme finite and separated over S so is the left hand side. □

Lemma 3. *There is a surjective comparison map* $c : \overline{M}_{g,\mathbf{w}}(\mathbb{P}^m, d) \to \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$.

Proof. This is similar to [22, Lemma 2.23]. □

Lemma 4. *For* $\mathbf{w}' \le \mathbf{w}$ *there is a surjective reduction morphism*

$$\rho_{\mathbf{w}'\mathbf{w}} : \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d) \to \overline{Q}_{g,\mathbf{w}'}(\mathbb{P}^m, d).$$

Proof. This follows from Lemma 3 and the corresponding result for stable maps (see [3, Proposition 1.2.1]). □

Since $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ is proper for $\mathbf{w} = 1^n$ the preceding lemma implies that $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ is proper in general.

[4] Actually $(m+1)(d-d')$

3. Localization

The virtual localization formulas [9] for $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ and $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ are the main tool we use to derive stable quotient relations. The existence of the necessary virtual fundamental class $[\overline{M}_{g,\mathbf{w}}(\mathbb{P}^m, d)]^{vir}$ for $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ has been shown in [1] and [3]. For the existence of a 2-term obstruction theory of $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^m, d)$ the same arguments as in [14, Section 3.2] can be used. They depend on existence of a ν-relative 2-term obstruction theory of the Quot scheme and the non-singularity of the Hassett moduli spaces of weighted curves. We will now follow Section 7 of [14].

In this paper we will only look at torus actions on moduli spaces of stable quotients or stable maps of $\mathbb{P}^1$ which are induced from the diagonal action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $([x_0 : x_1], \lambda) \mapsto [x_0 : \lambda x_1]$. By s we will denote the pull-back of the equivariant class $s \in A^1_{\mathbb{C}^*}(\mathrm{pt})$ defined as the first equivariant Chern class of the trivial rank 1 bundle on a point space with weight one action of $\mathbb{C}^*$ on it.

The equivariant cohomology of $\mathbb{P}^1$ is generated as an algebra over $\mathbb{Q}[s]$ by the equivariant classes $[0]$, $[\infty]$ of the two fixed points 0 and ∞. These classes satisfy (among others) the relation $[0] - [\infty] = s$. Localizing by s, the classes $[0]$ and $[\infty]$ give a basis of $A^*_{\mathbb{C}^*}(\mathbb{P}^1)$ as a free $\mathbb{Q}[s, s^{-1}]$-module.

3.1. Fixed loci.

The fixed loci for the action of $\mathbb{C}^*$ on $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ and $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ are very similar. They are both parametrized by the data of

(1) a graph $\Gamma = (V, E)$,
(2) a coloring $\zeta : V \to \{\pm 1\}$,
(3) a genus assignment $g : V \to \mathbb{Z}_{\geq 0}$,
(4) a map $d : V \sqcup E \to \mathbb{Z}_{\geq 0}$,
(5) a point assignment $p : \{1, \ldots, n\} \to V$,

such that Γ is connected and contains no self-edges, two vertices directly connected by an edge do not have the same color ζ,

$$g = h^1(\Gamma) + \sum_{v \in V} g(v),$$

$$d = \sum_{i \in V \sqcup E} d(i), \quad d|_E \geq 1,$$

and one further condition which depends on whether we look at $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ or $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$.

To state the stable quotient condition we need the following definition: A vertex $v \in V$ is called non-degenerate if the inequality

$$2g(v) - 2 + n(v) + \varepsilon d(v) > 0$$

holds for any $\varepsilon > 0$. Here $n(v)$ is the number of edges at v plus the number of preimages under p weighted by the corresponding weight.

Then for the combinatorial data on the stable quotients side we demand each vertex to be non-degenerate, whereas for the stable maps data the additional condition is $d|_V = 0$.

In the combinatorial data for $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ the vertices v of Γ correspond to curve components contracted to the fixed point of $\mathbb{P}^1$ specified by ζ, that is 0 for $\zeta(v) = 1$ and ∞ for $\zeta(v) = -1$. The edges correspond to multiple covers of $\mathbb{P}^1$ ramified only at 0 and ∞ with degree determined by the map d. For $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ the vertices of Γ correspond to components C of the curve over which the subsheaf S is an ideal sheaf of the trivial subsheaf given by one of the two[5] 1-dimensional fixed subspaces of $\mathbb{C}^2$, ζ specifies which subspace of $\mathbb{C}^2$ was chosen and $-d$ the degree of S. Edges correspond to multiple covers of $\mathbb{P}^1$ ramified in the two torus-fixed points.

The fixed locus corresponding to the combinatorial data is, up to a finite map, isomorphic to the product

$$\prod_{v\in V} \overline{M}_{g(v),(\mathbf{w}(v),\varepsilon^{d(v)})}/S_{d(v)},$$

where $\mathbf{w}(v)$ is a $(|p^{-1}(v)| + |\{e \text{ adjacent to } v\}|)$-tuple, which we will index by $p^{-1}(v) \sqcup \{e \text{ adjacent to } v\}$, such that $w(v)_i = w_i$ if $i \in p^{-1}(v)$ and $w(v)_e = 1$ for adjacent edges e. The symmetric group $S_{d(v)}$ permutes the ε-stable points. The product should be taken only over all non-degenerate vertices.

3.2. **The formula.** The virtual localization formula expresses the virtual fundamental class as a sum of the (virtual) fundamental classes of the fixed loci X weighted by the inverse of the equivariant Euler class of the corresponding virtual normal bundle N_X. In order to make sense of this inverse it is necessary to localize the equivariant cohomology ring by s.

The inverse of the Euler class of N_X is in both cases a product

$$\prod_{v} \mathrm{Cont}(v) \prod_{e \text{ edge}} \mathrm{Cont}(e),$$

for certain vertex and edge contributions depending only on the data of the graph corresponding to the vertex or edge. The contributions $\mathrm{Cont}(e)$ and $\mathrm{Cont}(v)$ for v degenerate are pulled back from the equivariant cohomology of a point. We will not need to know the exact form of the edge and unstable vertex contributions here apart from the fact that the contribution of a vertex v with $d(v) = g(v) = 0$ and $|\mathbf{w}(v)| = 1$ is equal to 1.

The non-degenerate vertex contribution is pulled back from

$$A^*_{\mathbb{C}^*}(\overline{M}_{g(v),(\mathbf{w}(v),\varepsilon^{d(v)})}/S_{d(v)}) \otimes \mathbb{Q}[s, s^{-1}].$$

It is given by

$$\mathrm{Cont}(v) = (\zeta(v)s)^{g(v)-d(v)-1} \sum_{j=0}^{\infty} \frac{c_i(\mathbb{F}_{d(v)})}{(\zeta(v)s)^i} \prod_e \frac{\zeta(v)s}{\omega_e^v - \psi_e}, \tag{1}$$

where $\mathbb{F}_d$ is the K-theory class $\mathbb{F}_d = \mathbb{E}^* - \mathbb{B}_d - \mathbb{C}$, the product is over edges e adjacent to v and

$$\omega_e^v = \frac{\zeta(v)s}{d(e)}.$$

The dual $\mathbb{E}^*$ of the Hodge bundle $\mathbb{E}$ has fiber

$$(H^0(C, \omega_C))^*$$

over a marked curve (C, p_i), the rank d bundle $\mathbb{B}_d$ on $\overline{M}_{g,(\mathbf{w},\varepsilon^d)}/S_d$ has the fiber

$$H^0(C, \mathcal{O}_C(p_{n+1} + \cdots + p_{n+d})|_{p_{n+1}+\cdots+p_{n+d}})$$

[5]These correspond to the two fixed points in $\mathbb{P}^1$.

over a marked curve (C, p_i). The bundle $\mathbb{B}_d$ can also be thought as an S_d invariant bundle living on $\overline{M}_{g,(\mathbf{w},\varepsilon^d)}$. In [14] the Chern classes of this lifted bundle have been computed:

$$c(-\mathbb{B}_d) = \prod_{i=n+1}^{n+d} \frac{1}{1 - \psi_{n+i} - \sum_{j=n+1}^{n+i} D_{ij}}.$$

Notice the similarity to the Φ function. The Chern classes for $\mathbb{B}_d$ on $\overline{M}_{g,(\mathbf{w},\varepsilon^d)}/S_d$ can be calculated by push-forward along the finite projection map and dividing by the degree $d!$.

3.3. **Comparison.** The $\mathbb{C}^*$ actions on $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ and $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ are compatible with the comparison map $c : \overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d) \to \overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ and c hence maps fixed loci to fixed loci. The image of a locus corresponding to some combinatorial data is given by contracting all degenerate vertices, adding the values of d of the contracted vertices and edges to the degree of the image vertex of the contraction, and replacing the point assignment p by its composition with the contraction.

The Gromov-Witten stable quotient comparison says that the contribution of a stable quotient fixed locus to the virtual fundamental class of $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)$ is the sum of push-forwards of the contributions of all stable maps loci in the preimage of c to the virtual fundamental class of $\overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d)$. This in particular implies that

$$c_*([\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1, d)]^{vir}) = [\overline{M}_{g,\mathbf{w}}(\mathbb{P}^1, d)]^{vir}.$$

4. The open locus

We will first restrict ourselves to the special case of n-fold tensor powers $M_{g|n}$ of the universal curve C_g over M_g. This case occurs when the weights are sufficiently small (i.e. $\sum_{i=1}^{n} w_i < 1$) and we restrict ourselves to the locus corresponding to smooth curves.

4.1. **Statement of the stable quotient relations.** Let us define the bracket operator $\{\}_\Delta$ on $\mathbb{Q}[\![t, p_1, \ldots, p_n]\!]$ in terms of the operators from the introduction by

$$\begin{aligned} \{f(t)\}_\Delta &= - \{f(t)\}_\kappa \\ \left\{ f(t) \prod_{i\in S} p_i \right\}_\Delta &= \{f(t)\}_{D_S} \\ \{p_i^e f(t, p_1, \ldots, p_n)\}_\Delta &= \{p_i f(t, p_1, \ldots, p_n)\}_\Delta, \end{aligned}$$

if $e > 0$. For example $\{t^2 p_1^3\}_\Delta = t^2\psi_1^2$ and $\{t^2 p_1 p_2\}_\Delta = -D_{12}\psi_1 = -D_{12}\psi_2$.

Proposition 2. *The relations given by*

$$\sum_{\zeta\in\{\pm 1\}} \zeta^{g-1} \left[\exp\left(-\frac{1}{2}\zeta p + \{\exp(-pD)\gamma(\zeta t, x)\}_\Delta \right) \right]_{t^r x^d \mathbf{p}^{\mathbf{a}}} = 0,$$

with the differential operator $D = tx\frac{\mathrm{d}}{\mathrm{d}x}$ and $p = p_1 + \cdots + p_n$ hold in $M_{g|n}$ under the condition $g - 2d - 1 + |\mathbf{a}| < r$.

4.2. **Proof of the stable quotient relations.** We generalize here the localization method of [19].

For each $i \in \{1, \dots, n\}$ we can define a class $s_i \in A^1(M_{g|n}(\mathbb{P}^1, d))$ as the pull-back of $c_1(\mathcal{S})$ from the universal curve $C_{g|n}(\mathbb{P}^1, d)$ via the i-th section.

For given nonnegative integers a_i let us look at the class

$$\mathbf{s}^{\mathbf{a}} = \prod_{i=1}^{n} (-s_i)^{a_i} \in A^{|\mathbf{a}|}(M_{g|n}(\mathbb{P}^1, d)).$$

The strategy is to study the $\mathbb{C}^*$ action from Section 3, to lift $\mathbf{s}^{\mathbf{a}}$ to equivariant cohomology and to write down the localization formula calculating the push-forward of this class to the equivariant cohomology of $M_{g|n}$. As we have seen the general form of the localization formula is a sum of contributions from the fixed loci. For each fixed locus a class from its equivariant Chow ring *localized by the localization variable* s pushed forward via the inclusion of the fixed locus. We get the relations from the fact that the rational functions in the localization variable we obtain must actually be polynomials in s after summing over all fixed loci.

Remark 1. In [14] and [19] the same strategy was pursued but other related classes were chosen to be pushed forward. In similar spirit we could add factors of the form $\pi_*(s_{n+1} c_1(\omega_\pi)^b)$, where $\pi : Q_{g|n+1}(\mathbb{P}^1, d) \to Q_{g|n}(\mathbb{P}^1, d)$ is the forgetful map and ω_π is the relative dualising sheaf, to the class we are pushing forward. However because of the commutative diagram

$$\begin{tikzcd} Q_{g|n+1}(\mathbb{P}^1, d) \arrow[r, "\pi"] \arrow[d, "\nu"] & Q_{g|n}(\mathbb{P}^1, d) \arrow[d, "\nu"] \\ M_{g|n+1} \arrow[r, "\pi"] & M_{g|n} \end{tikzcd}$$

and the fact that $c_1(\omega_\pi) = \nu^*(\psi_{n+1})$ these will be contained in the completed set of stable quotient relations.[6]

In [19] only factors with $a = 1$ were used to derive the FZ relations on M_g. The results from Section 4.3.2 imply that allowing higher values for a would not have led to more stable quotient relations.

The action of $\mathbb{C}^*$ on $\mathbb{P}^1$ is induced by the action of $\mathbb{C}^*$ on $\mathbb{C}^2$ given by $([z_0, z_1], \lambda) \mapsto [z_0, \lambda z_1]$. This naturally induces $\mathbb{C}^*$ actions not only on $Q_{g|n}(\mathbb{P}^1, d)$ but also on the universal curve $C_{g|n}(\mathbb{P}^1, d)$ and the universal sheaf $\mathcal{S}$. This gives a natural lift of the s_i to equivariant cohomology and therefore also a natural lift of $\mathbf{s}^{\mathbf{a}}$. We will not choose this lift but instead

$$\tilde{\mathbf{s}}^{\mathbf{a}} = \prod_{i=1}^{n} \left(-s_i - \frac{1}{2}s\right)^{a_i} \in A^{|\mathbf{a}|}_{\mathbb{C}^*}(M_{g|n}(\mathbb{P}^1, d)),$$

where the s_i are the natural lifts.

Let us consider the localization formula for this equivariant lift applied to the push-forward

$$\nu_*(\mathbf{s}^{\mathbf{a}} \cap [M_{g|n}(\mathbb{P}^1, d)]^{vir}) \in A_{2g+2d-2+n-|\mathbf{a}|}(M_{g|n}).$$

[6]To see this for more than one factor (say m factors) one needs to first interpret the product as a class on the m-fold tensor power X of $Q_{g|n+1}(\mathbb{P}^1, d)$ over $Q_{g|n}(\mathbb{P}^1, d)$ and use the birational map from $Q_{g|n+m}(\mathbb{P}^1, d)$ to X.

Let us shorten this by writing $\nu_*^{vir}(\mathbf{s}^{\mathbf{a}})$ for this push-forward after capping with the virtual fundamental class.

Because we assume the strict inequality $\sum_{i=1}^n w_i < 1$ there are only two fixed loci with respect to the torus actions in the description of Section 3.1. Concretely in this special case they correspond to elements $(C, p_1, \ldots, p_n)$ of $M_{g|n}$ with quotient sequence

$$0 \to \mathcal{O}_C(-p_{n+1} - \cdots - p_{n+d}) \to \mathcal{O}_C^2 \to Q \to 0$$

where the first map factors through $\mathcal{O}_C$ and the map $\mathcal{O}_C \to \mathcal{O}_C^2$ is induced from one of the two torus invariant inclusions of $\mathbb{C}$ as a coordinate axis in $\mathbb{C}^2$. Here $\mathcal{O}_C(-p_{n+1} - \cdots - p_{n+d})$ is an ideal sheaf of $\mathcal{O}_C$ of degree d. Both fixed point loci can be identified with $M_{g|n+d}/S_d$ where the symmetric group S_d permutes the last d markings.

Since the graphs corresponding to the fixed point loci have each only one vertex and no edge we can calculate the inverse of the equivariant Euler class of the normal bundle to each fixed locus using (1). It is given by

$$(\zeta s)^{g-d-1} \sum_{j=0}^{\infty} \frac{c_i(\mathbb{F}_d)}{(\zeta s)^i},$$

where ζ is $+1$ and -1 for 0 and ∞ respectively.

Applying the fixed point formula we obtain for the s^c part of the push-forward (2)

$$[\nu_*^{vir}(\tilde{\mathbf{s}}^{\mathbf{a}})]_{s^c} = \frac{1}{d!} \sum_{\zeta} \varepsilon_* \left[\prod_{i=1}^n \left(-s_i t - \frac{1}{2}\zeta \right)^{a_i} \sum_{j=0}^{\infty} t^j \zeta^{g-d-1-j} c_j(\mathbb{F}_d) \right]_{t^{g-1-d-c+|\mathbf{a}|}},$$

where $\varepsilon : M_{g|n+d} \to M_{g|n}$ forgets the last d markings. We have here by abuse of notation denoted similarly defined classes on $M_{g|n+d}$ with the same name as on $M_{g|n}(\mathbb{P}^1, d)$. The expression $(-s_i t - \frac{1}{2}\zeta)$ comes from the fact that the torus acts trivially on the subspace of $\mathbb{C}^2$ given by 0 and with weight 1 on the subspace given by ∞. The equivariant lift of the s_i was chosen in order to have this symmetric expression.

Since the push-forward must be an honest equivariant class, the classes (2) must be zero if $c < 0$.

Let us package these relations into a power series. We have

$$\prod_{i=1}^n \frac{1}{a_i!} [\nu_*^{vir}(\tilde{\mathbf{s}}^{\mathbf{a}})]_{s^c} = \sum_{\zeta} \zeta^{g-d-1} \varepsilon_* [\exp(-T_1) T_2]_{t^r x^d p^{\mathbf{a}}}, \tag{3}$$

with

$$T_1 = \sum_{i=1}^n \left(s_i t + \frac{1}{2}\zeta \right) p_i,$$

$$T_2 = \sum_{j=0}^{\infty} \sum_{d=0}^{\infty} (\zeta t)^j c_j(\mathbb{F}_d) \frac{x^d}{d!}.$$

Since

$$s_i = \sum_{j=n+1}^{n+d} D_{ij}$$

we can rewrite T_1 as

$$T_1 = \sum_{i=1}^{n} \sum_{j=n+1}^{n+d} D_{ij} p_i + \frac{1}{2} \zeta \sum_{i=1}^{n} p_i.$$

4.2.1. *Relations between diagonal classes.* In order to better understand $c(\mathbb{F}_d)$ let us here collect the universal relations between classes in $A^*(M_{g|n})$.

The basic relations are

$$\begin{aligned} D_{ij}\psi_i = D_{ij}\psi_j &= -D_{ij}^2 \\ D_{ij}D_{ik} &= D_{ijk} \end{aligned}$$

for pairwise different i, j, k (compare to [5]). Let $D_{S,a} \in A^a(M_{g|n})$ by defined for any $S \subset \{1, \dots, n\}$ and $a \geq |S| - 1$ by

$$D_{S,a} = \begin{cases} \psi_i^a, & \text{if } S = \{i\}, \\ (-1)^{|S|-1} D_S \psi_i^{a-|S|+1}, & \text{else, where } i \in \{1, \dots, d\}. \end{cases}$$

Then $D_{S,a} D_{T,b} = D_{S \cup T, a+b}$ if $S \cap T \neq \emptyset$.

Lemma 5. *Each monomial M in diagonal and cotangent line classes in $A^*(M_{g|n})$ can be written as a product*

$$M = \pm \prod_{S \in P} D_{S, a(S)+|S|-1}$$

for some partition P of $\{1, \dots, d\}$, function $a : P \to \mathbb{Z}_{\geq 0}$ and a suitable choice of sign. This product decomposition is unique if we only use the above relations between diagonal and cotangent line classes.

Remark 2. If the partition P is the one element set partition, we say that M is connected. We call the factors of the decomposition (or just the elements of P) the connected components of M.

The push-forward under the forgetful map $\pi : M_{g|n} \to M_{g|n-1}$ is given by

$$\pi_* D_{S,a} = \begin{cases} 0, & \text{if } n \notin S, \\ \kappa_{a-1}, & \text{if } |S \cap \{n\}| = 1, \\ -D_{S \setminus \{n\}, a-1}, & \text{else.} \end{cases}$$

Here and in the rest of this article κ_{-1} is defined to be zero.

4.2.2. *Simplest relations.* Let us first consider the stable quotient relations in the case of $\mathbf{a} = 0$. Then they are simply

$$0 = \sum_{\zeta} \zeta^{g-d-1} \varepsilon_* \left[\sum_{d=0}^{\infty} \sum_{j=0}^{\infty} (\zeta t)^j c_j(\mathbb{F}_d) \frac{x^d}{d!} \right]_{t^r x^d}$$

for

$$r > g - 1 - 2d. \tag{4}$$

By the definition of $\mathbb{F}_d$ as a K-theoretic difference of $\mathbb{E}^*$, a trivial rank 1 bundle and $\mathbb{B}_d$ the inner sum breaks into two factors. The part corresponding to $\mathbb{E}^*$ is pulled back from $M_{g|n}$ and does not depend on d. Using Mumford's formula [15] we can therefore rewrite the relations as: The class

$$\sum_{\zeta} \zeta^{g-d-1} \left[\exp\Big(-\sum_{i\geq 1} \frac{B_{2i}}{2i(2i-1)} \kappa_{2i-1}(t\zeta)^{2i-1} \Big) \varepsilon_* \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} (\zeta t)^j c_j(-\mathbb{B}_d) \frac{x^d}{d!} \right]_{t^r x^d}$$

is zero if (4) holds.

To deal with the second factor we formally expand it as a power series in x, t and the classes $D_{S,a}$ for various $S \subset \{n+1, \dots, n+d\}$ and $a \in \mathbb{Z}_{\geq 0}$. By the exponential formula and the facts from Section 4.2.1 we have that

$$\varepsilon_* \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} (\zeta t)^j c_j(-\mathbb{B}_d) \frac{x^d}{d!} = \exp \left(\varepsilon_* \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} S_d^r D_{\{n+1,\dots,n+d\},r} (\zeta t)^r \frac{x^d}{d!} \right)$$

where

$$\log \left(\sum_{d=0}^{\infty} \prod_{i=1}^{d} \frac{1}{1-it} \frac{x^d}{d!} \right) = \sum_{d=1}^{\infty} \sum_{r=0}^{\infty} S_d^r t^r \frac{x^d}{d!}.$$

With $\varepsilon_* D_{\{n+1,\dots,n+d\},r} = (-1)^{d-1} \kappa_{r-d}$ and noticing the similarities between the series defining S_d^r and $\log(\Phi)$ we obtain

$$\varepsilon_* \sum_{d=0}^{\infty} \zeta^d \sum_{j=0}^{\infty} \frac{(\zeta t)^j}{t^d} c_j(-\mathbb{B}_d) \frac{x^d}{d!} = \exp\left(-\{\log(\Phi(\zeta t))\}_\kappa\right)$$

and so the stable quotient relations in the case $\mathbf{a} = 0$ are

$$\left[\sum_{\zeta} \zeta^{g-1} \exp\left(-\{\gamma(\zeta t)\}_\kappa\right) \right]_{t^r x^d} = 0$$

under Condition (4).

4.2.3. *General relations.* We will investigate how monomials in the s_i affect the push-forward under ε of monomials supported only on the last d points.

Notice that for each partition P of $\{n+1, \dots, n+d\}$ we have

$$\exp\left(-\sum_{i=1}^{n} s_i \right) = \prod_{S \in P} \exp\left(-\sum_{i=1}^{n} \sum_{j \in S} D_{ij} \right), \tag{5}$$

and each factor is pulled back via the map forgetting all points in $\{n+1, \dots, n+d\}$ not in S.

Moreover notice that if M is a connected monomial supported in the last d marked points with $\varepsilon_* M = [-\{f(t)\}_\kappa]_{t^r} = [\{f(t)\}_\Delta]_{t^r}$, we have

$$[\varepsilon_*((-s_i p_i)^e x^d M)]_{x^d} = [\varepsilon_*((-dD_{i,n+1} p_i)^e x^d M)]_{x^d} = [\{(p_i D)^e (x^d f(t))\}_\Delta]_{t^r x^d}. \tag{6}$$

From (5), (6) and the identity

$$\exp(pD)\exp(f(t,x)) = \exp(\exp(pD) f(t,x))$$

we obtain the general stable quotient relations

$$\sum_{\zeta} \zeta^{g-1} \left[\exp\left(-\frac{1}{2}\zeta p + \{\exp(pD)\gamma(\zeta t, x)\}_\Delta \right) \right]_{t^r x^d \mathbf{p}^{\mathbf{a}}} = 0,$$

if

$$r > g - 1 - 2d + |\mathbf{a}|. \tag{7}$$

4.3. **Evaluation of the relations.**

4.3.1. *Minor simplification.* Notice that in the relations of Proposition 2 the summands in the ζ sum are equal up to a sign

$$(-1)^{g-1+r+|\mathbf{a}|}.$$

Therefore the relations are in fact zero if $g + r + |\mathbf{a}| \equiv 0 \pmod 2$, and in the case

$$g - 1 + r + |\mathbf{a}| \equiv 0 \pmod 2 \tag{8}$$

we can reformulate them to

$$\left[\exp\left(-\frac{1}{2}p + \{\exp(pD)\gamma(t,x)\}_\Delta\right)\right]_{t^r x^d \mathbf{p}^\mathbf{a}} = 0.$$

4.3.2. *Differential algebra.* In this section we will establish that it is enough to consider the stable quotient relations in the case that $a_i < 2$ for all $i \in \{1, \dots, m\}$.

The series $\delta = D\gamma - \frac{1}{2}$ satisfies the differential equation

$$D\delta = -\delta^2 + x + \frac{1}{4},$$

as can be derived from the differential equation

$$D(\Phi - D\Phi) = -x\Phi,$$

which is satisfied by Φ as seen by looking at its series definition.

Reformulating the relations with δ gives

$$\left[\exp\left(-\{\gamma\}_\kappa + \sum_{i=1}^\infty \frac{1}{i!}\{p^i D^{i-1}\delta\}_\Delta\right)\right]_{t^r x^d \mathbf{p}^\mathbf{a}} = 0$$

if (7) and (8) hold.

Let us consider

$$G = \frac{\frac{\partial^2}{\partial p_j^2} F}{F},$$

with

$$F = \exp\left(-\{\gamma\}_\kappa + \sum_{i=1}^\infty \frac{1}{i!}\{p^i D^{i-1}\delta\}_\Delta\right).$$

We have that

$$G = \frac{\partial^2}{\partial p_j^2}\sum_{i=1}^\infty \frac{1}{i!}\{p^i D^{i-1}\delta\}_\Delta + \left(\frac{\partial}{\partial p_j}\sum_{i=1}^\infty \frac{1}{i!}\{p^i D^{i-1}\delta\}_\Delta\right)^2$$
$$= \{\exp(pD)D\delta\}_{\Delta_j} + \{\exp(pD)\delta\}_{\Delta_j}^2,$$

where the operator $\{\}_{\Delta_j}$ is defined by $\{f\}_{\Delta_j} = p_j^{-1}\{p_j f\}_\Delta$. The bracket $\{\}_{\Delta_j}$ and squaring commute because, informally said, $\{\}_{\Delta_j}$ connects any term to j. So we obtain

$$\begin{aligned} G =&\{\exp(pD)D\delta + (\exp(pD)\delta)^2\}_{\Delta_j} = \{\exp(pD)(D\delta + \delta^2)\}_{\Delta_j} \\ =&\left\{\exp(pD)\left(x + \frac{1}{4}\right)\right\}_{\Delta_j} = \frac{1}{4} + x\{\exp(pt)\}_{\Delta_j}, \end{aligned}$$

Therefore we have that

$$\frac{\partial^2}{\partial p_j^2} F = \left(\frac{1}{4} + x\{\exp(pt)\}_{\Delta_j}\right) F.$$

and can express the relations for $a_j \geq 2$ in terms of lower relations multiplied by monomials in cotangent line and diagonal classes.

4.3.3. *Substitution.* The differential equation satisfied by $-t\gamma$ has been studied by Ionel in [11]. In [19] her results were extended to give formulas for $D^i\gamma$. We will collect some of their results on γ and its derivatives here.

With the new variables

$$u = \frac{t}{\sqrt{1+4x}}, \qquad y = \frac{-x}{1+4x}$$

one can write

$$\gamma = \frac{1}{t}(t\gamma)(0,x) + \frac{1}{4}\log(1+4y) + \sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}u^k y^j,$$

for some coefficients $c_{k,j}$, which are defined to vanish outside the summation region used above. Furthermore, we have for the derivatives of δ

$$D^{i-1}\delta = (1+4y)^{-\frac{i}{2}}\left(\sum_{j=0}^{i-1} b_j^i u^{i-1} y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+i} c_{k,j}^i u^{k+i} y^j\right) =: (1+4y)^{-\frac{i}{2}}\delta_i,$$

for some coefficients b_j^i, $c_{k,j}^i$.

We will also need a result by Ionel relating coefficients of a power series F in x and t before and after the variable substitution:

$$[F]_{t^r x^d} = (-1)^d[(1+4y)^{\frac{r+2d-2}{2}}F]_{u^r y^d}$$

Let us now apply these formulas to the relations. Using the fact that $\kappa_{-1} = 0$ we get

$$\left[(1+4y)^e \exp\left(-\left\{\sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}u^k y^j\right\}_{\kappa} + \sum_{i=1}^{\infty}\frac{1}{i!}\{p^i\delta_i\}_{\Delta}\right)\right]_{u^r y^d \mathbf{p}^{\mathbf{a}}} = 0,$$

under conditions (7) and (8), where the exponent e is

$$e = \frac{r+2d-2}{2} - \frac{\kappa_0}{4} - \frac{|\mathbf{a}|}{2} = \frac{r+2d-1-g-|\mathbf{a}|}{2}.$$

4.3.4. *Extremal coefficients.* It is noticeable that in the series

$$\sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}u^k y^j, \quad \sum_{j=0}^{i-1} b_j^i u^{i-1} y^j - \sum_{k=0}^{\infty}\sum_{j=0}^{k+i} c_{k,j}^i u^{k+i} y^j$$

appearing in the formulas for γ and $D^{i-1}\delta$ the y-degree is bounded from above by the u-degree. We will only be interested in the extremal coefficients. Here the A and C_i come into the picture.

In [11] it has been proven that

$$\log(A(t)) = \sum_{k=1}^{\infty} c_{k,k} t^k$$

and in [19] it is shown that

$$2^{-i} C_i(t) = b_{i-1}^i t^{i-1} - \sum_{k=0}^{\infty} c_{k,k+i}^i t^{k+i}.$$

We see that in the exponential factor of the relations for each summand the y-degree is bounded from above by the u-degree. Furthermore the exponent e of the $(1+4y)$-factor is integral and positive by (8) and (7). This implies that the relation is zero unless

$$3r \geq g + 1 + |\mathbf{a}| \tag{9}$$

holds. With the following lemma we can extract the extremal part of the relations.

Lemma 6. *Fix any $\mathbb{Q}$ algebra A, any $F \in A[y]$ and any $c \in \mathbb{Z}_{\geq 0}$. The relations*

$$[(1+4y)^d F]_{y^d} = 0$$

for all $d > c$ imply $F = 0$.

Proof. The relations can be rewritten to

$$\left[\left(\frac{1}{y} + 4\right)^d F\right]_{y^0} = 0.$$

The lemma follows by the fact that F is a polynomial in y, and linear algebra. $\square$

Using the lemma and the formulas for the extremal coefficients we obtain that the relations

$$\left[\exp(-\{\log(A)\}_\kappa)\exp\left(\sum_{i=1}^{\infty}\frac{1}{i!}\{p^i C_i\}_\Delta\right)\right]_{z^r \mathbf{p}^{\mathbf{a}}} = 0$$

hold under the conditions (8) and (9).

Ignoring terms of higher order in the p_i the second factor can be rewritten

$$\begin{aligned}\exp\left(\sum_{i=1}^{\infty}\frac{1}{i!}\{p^i C_i\}_\Delta\right) &\equiv \exp\left(\sum_{\emptyset \neq S \subseteq \{1,\dots,n\}} \{p^S C_{|S|}\}_\Delta\right) \\ &\equiv \sum_{S \subseteq \{1,\dots,n\}} p^S \sum_{P \vdash S} \prod_{i \in P} \{C_{|i|}\}_{D_i}.\end{aligned}$$

Thus we finally obtain that the FZ-type relations

$$\left[\exp(-\{\log(A)\}_\kappa)\sum_{P\vdash S}\prod_{i\in P}\{C_{|i|}\}_{D_i}\right]_{z^r}=0,$$

hold for any $S\subseteq\{1,\dots,n\}$ if (8) and (9).

5. Localization sums

In the derivation of the more general stable quotient relations we will need to deal with two types of localization sums related to the nodes and marked points, respectively. To keep the length of the proof of the stable quotient relations more reasonable we will deal with them here. The sums are genus independent and have been studied more generally by Givental [6]. We apply here his method in a special case. See also [7] and [12].

Let $N^*_{\mathbb{C}^*}(\mathbb{P}^1)$ be the Novikov ring of $\mathbb{P}^1$ with values in $\mathbb{C}[s,s^{-1}]$. We define a formal Frobenius manifold structure on $X=A^*_{\mathbb{C}^*}(\mathbb{P}^1)\otimes N^*_{\mathbb{C}^*}(\mathbb{P}^1)$ over $N^*_{\mathbb{C}^*}(\mathbb{P}^1)$ using equivariant Gromov-Witten theory. The $N^*_{\mathbb{C}^*}(\mathbb{P}^1)$-module X is free of dimension two with basis $\{\phi_0,\phi_\infty\}$ for $\phi_i=[i]/e_i$, where e_i is the equivariant Euler class of the tangent space of $\mathbb{P}^1$ at i. We denote the corresponding coordinate functions by y_0, y_∞. This gives a basis $\{\frac{\partial}{\partial y_0},\frac{\partial}{\partial y_\infty}\}$ of the space of vector fields on X. The metric is given in this basis by

$$g=\begin{pmatrix}s^{-1}&0\\0&-s^{-1}\end{pmatrix}=\begin{pmatrix}e_0^{-1}&0\\0&e_\infty^{-1}\end{pmatrix}.$$

The primary equivariant Gromov-Witten potential is

$$\sum_{n=0}^\infty\frac{1}{n!}\sum_{d=0}^\infty x^d\int_{[\overline{M}_{0,n}(\mathbb{P}^1,d)]^{vir}}\prod_{i=1}^n\mathrm{ev}_i^*(y_0\phi_0+y_\infty\phi_\infty)=\frac{y_0^3}{6s}-\frac{y_\infty^3}{6s}+xe^w,$$

where we have set $w=(y_0-y_\infty)/s$ (compare to [17]). Thus we have

$$\alpha=\begin{pmatrix}\mathrm{d}y_0+\frac{x}{e_0}e^w\mathrm{d}w&\frac{x}{e_\infty}e^w\mathrm{d}w\\\frac{x}{e_0}e^w\mathrm{d}w&\mathrm{d}y_\infty+\frac{x}{e_\infty}e^w\mathrm{d}w\end{pmatrix},$$

for the matrix of one forms

$$\alpha=\sum_i A_i\mathrm{d}y_i,$$

where the matrices A_i are defined by the quantum product

$$\frac{\partial}{\partial y_i}\star\frac{\partial}{\partial y_a}=\sum_b[A_i]_a^b\frac{\partial}{\partial y_b}.$$

Using α we can compactly write down the differential equation for the S-matrix

$$(t\mathrm{d}-\alpha)S=0,$$

with initial condition $S(0,0)=\mathrm{Id}$.

If we set

$$S=\begin{pmatrix}S_0^0&S_\infty^0\\S_0^\infty&S_\infty^\infty\end{pmatrix},$$

this gives the system

$$t\mathrm{d}S_i^j=S_i^j\mathrm{d}y_j+\sum_k S_i^k\frac{x}{e_k}e^w\mathrm{d}w,$$

with unique solution

$$S_i^j = e^{\frac{y_i}{t}} \left(\left(\frac{1+\zeta_i\zeta_j}{2} - tx\frac{\mathrm{d}}{\mathrm{d}x} \right) \Phi \right) \left(-\frac{t}{e_i}, \frac{xe^w}{e_i^2} \right)$$

under the initial conditions. Here the signs $\zeta_i \in \{\pm 1\}$ are defined by $\zeta_i = \frac{e_i}{s} \in \{\pm 1\}$.

There is a set of canonical coordinates u^i on X defined by the localization sums

$$u^i = \sum_{n=0}^{\infty} \sum_{d=0}^{\infty} x^d \sum_{\Gamma \in G_{0,n+2}^{u^i}(\mathbb{P}^1, d)} \mathrm{Cont}_\Gamma \left(e_i \frac{\partial^2 F_0}{(\partial y_i)^2} \right),$$

where $G_{0,n+2}^{u^i}(\mathbb{P}^1, d)$ is the set of $(n+2)$-point (each of weight 1) degree d localization graphs with the first two points on a single component contracted to i and Cont_Γ stands for the contribution of a graph Γ in the localization formula. By the Gromov-Witten stable quotient comparison and since there is only one fixed locus on the stable quotient side we have

$$u^i(0,0) = e_i e_i^{-d-1} \sum_{d=1}^{\infty} \frac{x^d}{d!} \int_{\overline{M}_{0,2|d}} \sum_{j=0}^{\infty} \frac{c_j(\mathbb{F}_d)}{e_i^j},$$

where $\overline{M}_{0,2|d} = \overline{M}_{0,(1^2,\varepsilon^d)}$ is a Losev-Manin space. Since the Hodge bundle is trivial on $\overline{M}_{0,2|d}$ this simplifies to

$$u^i(0,0) = e_i e_i^{-d-1} \sum_{d=1}^{\infty} \frac{x^d}{d!} \int_{\overline{M}_{0,2|d}} \sum_{j=0}^{\infty} \frac{c_j(-\mathbb{B}_d)}{e_i^j}.$$

Recalling the definition of $\mathbb{B}_d$ we can write the integrand on the right hand side as a sum of monomials in ψ- and diagonal classes. The integral of such a monomial vanishes unless it is the diagonal where all d points come together. The constants $-C_d^{-1}$ which are defined from $\log(\Phi)$ by

$$\log(\Phi(t,x)) = \sum_{d=1}^{\infty} \sum_{r=-1}^{\infty} C_d^r t^r \frac{x^d}{d!}$$

exactly count these contributions. So we get

$$u^i(0,0) = -\sum_{d=1}^{\infty} C_d^{-1} \frac{x^d}{d!} e_i^{-2d+1}. \tag{10}$$

Using the S-matrix and the u^i we can now calculate the series we are interested in. We will not regard S and the u^i outside $(0,0)$, so let us write from now on S_i^j for $S_i^j(0,0)$ and u^i for $u^i(0,0)$.

The first series P^{ij} will be needed to deal with stable quotient localization chains containing one of the weight 1 marked points. We have

$$P^{ij}(t,x) := \frac{1+\zeta_i\zeta_j}{2} + \sum_{d=1}^{\infty} x^d \sum_{\Gamma \in G_{0,2}^{P^{ij}}(\mathbb{P}^1,d)} \frac{e_i e^{u^i/\omega_\Gamma}}{\omega_\Gamma - t} \mathrm{Cont}_\Gamma \left(\frac{\partial F_0}{\partial y_i \partial y_j} \right) = e^{u^i/t} \zeta_i \zeta_j S_i^j(-t),$$

where $G_{0,2}^{P^{ij}}(\mathbb{P}^1, d)$ is the set of all localization graphs with the first marking on a valence 2 vertex at i and the second marking at j and ω_Γ is the ω as in (1) corresponding to the flag at the first marking.

The second series is needed to deal with chains at the nodes of the curve. We have

$$E^{ij}(t_1,t_2,x) := \sum_{d=1}^{\infty} x^d \sum_{\Gamma\in G_{0,2}^{E^{ij}}(\mathbb{P}^1,d)} \frac{e_i e^{u^i/\omega_{\Gamma,1}}}{\omega_{\Gamma,1}-t_1} \frac{e_j e^{u^j/\omega_{\Gamma,2}}}{\omega_{\Gamma,2}-t_2} \operatorname{Cont}_\Gamma\left(\frac{\partial F_0}{\partial y_i \partial y_j}\right)$$
$$= \frac{s}{t_1+t_2}\left(\frac{\zeta_i+\zeta_j}{2} - e^{u^i/t_1+u^j/t_2}\zeta_i\zeta_j \sum_\mu S_i^\mu(-t_1)\zeta_\mu S_j^\mu(-t_2)\right),$$

where $G_{0,2}^{E^{ij}}(\mathbb{P}^1,d)$ is the set of all localization graphs with both markings on valence 2 vertices, the first marking mapped to i, the second marking to j, and $\omega_{\Gamma,1}$, $\omega_{\Gamma,2}$ are ω as in Section 3.2 corresponding to the flag at the first and second marking respectively.

Let us now simplify the expressions for $P^{ij}(t,x)$ and $E^{ij}(t_1,t_2,x)$ using the explicit S-matrix and canonical coordinates at 0. For $P^{ij}(t,x)$ we obtain

$$P^{ij}(t,x) = \exp\left(-\sum_{d=1}^{\infty} C_d^{-1} t^{-1}\frac{x^d}{d!}e_i^{-2d}\right)\left(\left(\frac{1+\zeta_i\zeta_j}{2} - t\zeta_i\zeta_j x\frac{\mathrm{d}}{\mathrm{d}x}\right)\Phi\right)\left(\frac{t}{e_i},\frac{x}{e_i^2}\right)$$
$$= \left(\left(\frac{1}{2}-\zeta_i\zeta_j\delta\right)\Phi'\right)\left(\frac{t}{e_i},\frac{x}{e_i^2}\right),$$

where Φ' is defined by

$$\Phi'(t,x) = \exp\left(-\sum_{d=1}^{\infty} C_d^{-1}t^{-1}\frac{x^d}{d!}\right)\Phi(t,x) = \exp\left(\sum_{d=1}^{\infty}\sum_{r=0}^{\infty} C_d^r t^r \frac{x^d}{d!}\right).$$

Similarly we obtain

$$\frac{t_1+t_2}{s}E^{ij}(t_1,t_2,x) = \frac{\zeta_i+\zeta_j}{2} + \Phi'\left(\frac{t_1}{e_i},\frac{x}{e_i^2}\right)\Phi'\left(\frac{t_2}{e_j},\frac{x}{e_j^2}\right)\left(\zeta_i\delta\left(\frac{t_1}{e_i},\frac{x}{e_i^2}\right)+\zeta_j\delta\left(\frac{t_2}{e_j},\frac{x}{e_j^2}\right)\right).$$

6. The general case

We now extend the relations of Section 4 to the boundary and allow nearly arbitrary weights. More precisely we will assume that:

$$\text{If for some } S\subset\{1,\dots,n\} \text{ we have } \sum_{i\in S} w_i = 1, \text{ we must have } |S|=1. \tag{11}$$

We still obtain relations for any weight data since one can always modify $\mathbf{w}$ a bit such that the moduli space $\overline{M}_{g,\mathbf{w}}$ does not change (whereas $\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1,d)$ will change). It is even not necessary to allow that there exist points of weight 1 at all but it is interesting to see different ways of obtaining the same relations.

6.1. **Statement of the stable quotient relations.** Let $\mathcal{G}$ be the set of stable graphs describing the strata on $\overline{M}_{g,\mathbf{w}}$. The data of $\mathcal{G}$ in particular includes a map p from $\{1,\dots,n\}$ to the set of vertices as in Section 3.1. Let us assume that the first n' points are of weight different from 1 and the remaining n'' points are of weight 1.

Proposition 3. *For a codimension r, a degree d, an n-tuple $\mathbf{a}$ such that $g-2d-1+|\mathbf{a}|<r-|E|$, in $A^r(\overline{M}_{g,\mathbf{w}})$ the relation*

$$0=\left[\sum_{\substack{\Gamma\in\mathcal{G}\\ \zeta:\Gamma\to\{\pm1\}}}\frac{1}{\mathrm{Aut}(\Gamma)}\xi_{\Gamma*}\Big(\prod_v \mathrm{Vertex}^{3\zeta(v)}_v\prod_e \mathrm{Edge}^{3\zeta(v_1),\zeta(v_2)}_e\right.$$
$$\left.\prod_{i=n'+1}^n \mathrm{Point}^3(\psi_i\zeta(p(i))t,p_i)\Big)\right]_{t^{r-|E|}x^d\mathbf{p}^{\mathbf{a}}}$$

holds, where $\mathrm{Vertex}^{3\zeta}_v$ *is a product*

$$\mathrm{Vertex}^{3\zeta}_v=\zeta^{g(v)-1}\exp\left(-\frac{1}{2}\zeta p_{(v)}+\{\exp(p_{(v)}D)\gamma(t\zeta,x)\}_{\Delta^{(v)}}+V^\zeta\right)$$

in $A^*(\overline{M}_{g(v),\mathbf{w}_v})[x,t,\mathbf{p}]$, where $p_{(v)}=\sum_{n'\geq i\in p^{-1}(v)}p_i$, $\{\}_{\Delta^{(v)}}$ is defined identically compared to $\{\}_\Delta$ but κ_j is replaced by $\kappa_j^{(v)}$ — the κ_j class at v — and

$$V^\zeta=-\sum_{i\geq1}\frac{B_{2i}}{2i\cdot(2i-1)}\sum_{\Delta\in\mathcal{DG}}\frac{1}{\mathrm{Aut}(\Delta)}\xi_{\Delta,*}\left(\frac{\psi_a^{2i-1}+\psi_b^{2i-1}}{\psi_a+\psi_b}\right)(\zeta t)^{2i-1}.$$

Here $\mathcal{DG}\subseteq\mathcal{G}$ is the set of graphs corresponding to divisor classes, i.e. graphs with only one edge, and ψ_a, ψ_b are the two cotangent line classes corresponding to the unique node corresponding to a divisor. The edge and point series are given by

$$\begin{aligned}t(\psi_1+\psi_2)\mathrm{Edge}^{3\zeta_1,\zeta_2}_e=&(\Phi'(t\zeta_1\psi_1)\Phi'(t\zeta_2\psi_2))^{-1}\frac{\zeta_1+\zeta_2}{2}+\\&\zeta_1\delta(t\zeta_1\psi_1,x)+\zeta_2\delta(t\zeta_2\psi_2,x),\end{aligned}$$

where Φ' is from Section 5, and

$$\mathrm{Point}^3(t,p)=1+p\delta(t,x).$$

6.2. **Proof of the stable quotient relations.** In Section 4.2 we looked at the integrand $\mathbf{s}^{\mathbf{b}}$ coming from powers of the pulled back first Chern class of the universal sheaf $\mathcal{S}$ over $C_{g|n}(\mathbb{P}^1,d)$. This works also well for $\overline{Q}_{g|\mathbf{w}}(\mathbb{P}^1,d)$ if no point is of weight 1. For the points of weight 1 we however need to choose different classes since by the stability conditions the torsion of $\mathcal{S}$ must be away from the sections of points of weight 1. Instead we will pull back classes from $\mathbb{P}^1$ via evaluation maps.

For an n-tuple $\mathbf{a}$, which can be split into an n'-tuple $\mathbf{a}'$ and an n''-tuple $\mathbf{a}''$, we will thus study the equivariant class

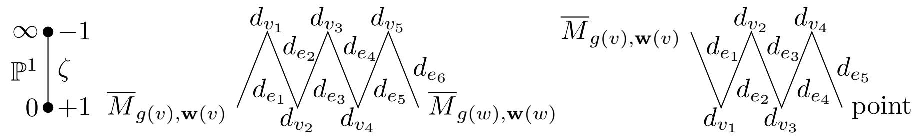
$$\varphi(\mathbf{a}):=\tilde{\mathbf{s}}^{\mathbf{a}'}\prod_{i=n'+1}^n \mathrm{ev}_i^*\left(\left(\frac{[0]+[\infty]}{2}\right)^{a_i}\right)\in A^{|\mathbf{a}|}_{\mathbb{C}^*}(\overline{Q}_{g,\mathbf{w}}(\mathbb{P}^1,d)),$$

where $[0]$ and $[\infty]$ are the equivariant classes of 0 and ∞ respectively, and $\tilde{\mathbf{s}}^{\mathbf{a}'}$ is the equivariant class from Section 4.2. Since $A^*_{\mathbb{C}^*}(\mathbb{P}^1)\otimes\mathbb{Q}[s,s^{-1}]$ is a two dimensional $\mathbb{Q}[s,s^{-1}]$ module we do not lose any relations by considering only the case when $\mathbf{a}''$ is $\{0,1\}$ valued.

As before we consider the s^c part of the push-forward of $\varphi(\mathbf{a})$ for $c<0$ when using the virtual localization formula.

In order to gain overview over the in d monotonously growing number of fixed loci we sort them according to the stratum of $\overline{M}_{g,\mathbf{w}}$ they push forward to. Let us

Figure 1. An edge and a point chain



consider stable graphs $\Gamma = (V, E)$ of $\overline{M}_{g,\mathbf{w}}$ together with a coloring $\zeta : V \to \{\pm 1\}$ and a degree assignment $d : V \sqcup E \sqcup \{n'+1, \ldots, n\} \to \mathbb{Z}_{\geq 0}$. This data records coloring and the $d|_V$ from a stable quotient graph, the total degrees of the chains which destabilize to a node or a weight 1 marked point when pushing forward to $\overline{M}_{g,\mathbf{w}}$. To get back to a stable quotient graph one needs to choose for each edge and each weight 1 marked point of degree d a splitting $d = d_{e_1} + d_{v_1} + \cdots + d_{v_{\ell-1}} + d_{e_\ell}$ of d corresponding to the degrees on the chain (see Figure 1). Because of the conditions on the coloring of a stable graph there is a mod 2 condition on the length ℓ of the chains depending on the color at the connection vertices or $\mathbf{a}$. If we had not imposed (11), we would also have chains for each set of marked points of total weight exactly 1.

The fixed loci corresponding to such a decorated stable graph are, up to a finite map, isomorphic to products

$$\prod_v \overline{M}_{g(v),(\mathbf{w}(v),\varepsilon^{d(v)})}/S_{d(v)}$$

times a number of Losev-Manin factors $\overline{M}_{0,2|d}/S_d$ corresponding to the vertices inside the chains.

For the localization calculation we will also have to consider the pull-back of the integrand $\varphi(\mathbf{a})$ to each fixed locus. The factors $\mathrm{ev}_i^*(([0]+[\infty])^{a_i})$ of $\varphi(\mathbf{a})$ merely restrict the coloring of the stable quotient graphs and their contribution is pulled back from the equivariant cohomology of a point. The other factors $s_i^{a_i}$ need to be partitioned along the factors of the stable quotient fixed locus. However it is easy to see that in order for the contribution to be nonzero $s_i^{a_i}$ has to land at the factor corresponding to the vertex $p(i)$.

After all these preconsiderations let us write down the localization formula. It will be the most convenient to write it in a power series form. We have

$$\nu_*^{vir}\varphi(\mathbf{a}) = \left[\sum_{\Gamma,\zeta,d} \frac{1}{\mathrm{Aut}(\Gamma)} \xi_{\Gamma *}\varepsilon_* \left(\prod_v \mathrm{Vertex}_v^1 \prod_e \mathrm{Edge}_e^1 \prod_{i=n'+1}^{n} \mathrm{Point}_i^1\right)\right]_{x^d\mathbf{p}^{\mathbf{a}}},$$

where ν_*^{vir} again denotes push-forward using ν after capping with the virtual fundamental class and the vertex, edge and point series still need to be defined.

The vertex series Vertex_v^1 is given by

$$\mathrm{Vertex}_v^1 = (\zeta(v)s)^{g(v)-1}\exp(-T_1)T_2 \in \bigoplus_{d=0}^{\infty} A_{\mathbb{C}^*}^*(\overline{M}_{g(v),(\mathbf{w}(v),\varepsilon^d)}) \otimes \mathbb{Q}[s,s^{-1}][\![x,\mathbf{p}]\!]$$

with

$$T_1 = \sum_{n' \geq i \in p^{-1}(v)} s_i p_i + \frac{1}{2}\zeta(v) \sum_{n' \geq i \in p^{-1}(v)} s p_i$$
$$T_2 = \sum_{j=0}^{\infty} \sum_{d=0}^{\infty} (\zeta(v)s)^{d-j} c_j(\mathbb{F}_d) \frac{x^d}{(\zeta(v)s)^{2d} d!}.$$

For the edge series Edge_e^1 we have

$$\mathrm{Edge}_e^1 = \sum_{d=1}^{\infty} x^d \sum_{\Gamma_e^d = (V_e, E_e)} \frac{1}{\mathrm{Aut}(\Gamma_e^d)} \frac{\zeta_1 s}{\omega_{\Gamma_e^d,1} - \psi_1} \frac{\zeta_2 s}{\omega_{\Gamma_e^d,2} - \psi_2} \prod_{f \text{ edge}} \mathrm{Cont}(f) \prod_{v \text{ vertex}} \mathrm{Cont}(v),$$

where Γ_e^d is a stable quotient localization graph of $\overline{Q}_{0,2}(\mathbb{P}^1, d)$ [7] with color of the two vertices determined by ζ at the two vertices adjacent to e. The variables ζ_i, d_i, ψ_i for $i \in \{1, 2\}$ denote the color, the weight and the ψ-classes of the two marked points corresponding to the edge respectively. The contributions $\mathrm{Cont}(f)$ and $\mathrm{Cont}(v)$ are contributions to the calculation of the integrals

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^1,d)} \mathrm{ev}_1^*(\phi_1) \, \mathrm{ev}_2^*(\phi_2)$$

as in Section 5. By the stable quotients Gromov-Witten comparison we can replace $\overline{Q}_{0,2}(\mathbb{P}^1, d)$ everywhere in this paragraph by $\overline{M}_{0,2}(\mathbb{P}^1, d)$ and then Edge_e^1 becomes very similar to $E^{ij}(t_1, t_2, x)$ from Section 5.

Finally the point series Point_i^1 is similarly given by

$$\mathrm{Point}_i^1 = \left(\frac{s}{2}\zeta_i\right)^{a_i} + \sum_{d=1}^{\infty} x^d \sum_{\Gamma_i^d = (V_i, E_i)} \frac{1}{\mathrm{Aut}(\Gamma_i^d)} \frac{\zeta_i s}{\omega_{\Gamma_i^d} - \psi_i} \prod_{f \text{ edge}} \mathrm{Cont}(f) \prod_{v \text{ vertex}} \mathrm{Cont}(v),$$

where ζ_i is ζ at the vertex with i, Γ_i^d is a stable quotient localization graph of $\overline{Q}_{0,2}(\mathbb{P}^1, d)$ with color of the first vertex determined by ζ_i. Here the contributions $\mathrm{Cont}(f)$ and $\mathrm{Cont}(v)$ are contributions to the calculation of the integrals

$$\int_{\overline{Q}_{0,2}(\mathbb{P}^1,d)} \mathrm{ev}_1^*(\phi_1) \, \mathrm{ev}_2^*\left(\phi_2 \left(\frac{[0] + [\infty]}{2}\right)^{a_i}\right).$$

The first summand corresponds to the case when the length of the chain is 0. Its form comes from the identity $[0] - [\infty] = s$ and the fact that $[0] \cdot [\infty] = 0$.

6.2.1. *Pushing forward.* Next we will study the ε-push-forward in the formula for $\nu_*^{vir} \varphi(\mathbf{a})$. For this we want to replace the ψ-classes in the edge and point terms by ψ-classes pulled back via ε, since the other ψ-classes are already pull-backs. For a fixed localization graph ε is a composition of local maps, one for each vertex in

[7] which is here a chain similar to the first in Figure 1

the graph, of the form $\overline{M}_{g(v),(w(v),\varepsilon^{d(v)})} \to \overline{M}_{g(v),w(v)}$, and one for each edge and marked point which contracts products of factors of the form $\overline{M}_{0,2|d}$.

Let us first look at the push-forward local to a vertex. For this we only need to look at the product of the vertex term with the factors of the form

$$\frac{1}{\omega_i - \psi_i}$$

from the adjacent edge and point series. Let us simplify the notation for this local discussion. With d we denote the degree at this vertex, with $n = n' + n''$ the number of marked points with weight different or equal to 1 respectively, and with $\mathbf{w}$ the weights at the vertex. We will index the marked points by the set $\{1, \ldots, n\} \sqcup \{1, \ldots, d\}$. Hopefully the non-empty intersection of these sets will not cause any confusion.

The basic pull-back formula is that

$$\psi_i = \varepsilon^*(\psi_i) + \Delta_{i1}$$

in the case that $d = 1$. Here Δ_{i1} is the boundary divisor of curves who have one irreducible component of genus 0 containing only i and the weight ε point. This generalizes to the formula

$$\psi_i^k = \varepsilon^*(\psi_i^k) + \varepsilon^*(\psi_i^{k-1}) \sum_{\emptyset \neq T \subset \{1,\ldots,d\}} \Delta_{iT},$$

where Δ_{iT} is the boundary divisor of curves who have one genus 0 irreducible component containing only i and the weight ε points indexed by T. Thus

$$\frac{1}{\omega_i - \psi_i} = \frac{1}{\omega_i - \varepsilon^*(\psi_i)} \left(1 + \omega_i^{-1} \sum_{\emptyset \neq T \subset \{1,\ldots,d\}} \Delta_{iT} \right).$$

Modulo factors pulled back via ε the most general classes we will need to push forward are products of factors of the form

- D_{iS} for $i \in \{1, \ldots, n'\}$, $S \subset \{1, \ldots, d\}$,
- Δ_{iT} for $i \in \{n'+1, \ldots, n\}$, $T \subset \{1, \ldots, d\}$,
- M, a monomial in ψ- and diagonal classes of the d points of weight ε.

In order for the push-forward to be nonzero the sets S and T must be pairwise disjoint. Moreover for each factor Δ_{iT}, the diagonal class corresponding to T must be a connected factor of M.

Therefore for a monomial $M = \prod_{i \in P} M_i$ corresponding to a set partition $P \vdash \{1, \ldots, d\}$ we obtain

$$\varepsilon_* \left(\prod_{i=n'+1}^{n} \left(1 + \sum_{\emptyset \neq T \subset \{1,\ldots,d\}} \omega_i^{-1} \Delta_{iT} \right) \exp\left(\sum_{i=1}^{n'} s_i p_i \right) M \right) =$$

$$\prod_{i \in P} \left(\delta_i^{\Delta} \sum_{j=n'+1}^{n} \omega_j^{-1} + \varepsilon_{i*} \left(\exp\left(\sum_{j=1}^{n'} s_j p_j \right) M_i \right) \right),$$

where here δ_i^Δ is one if M_i is a diagonal class and zero otherwise and the ε_i are forgetful maps $\varepsilon_i : \overline{M}_{g,(\mathbf{w},\varepsilon^{|i|})} \to \overline{M}_{g,\mathbf{w}}$ [8]. Notice that the push-forward is a product where each factor is dependent only on one factor of M.

This allows us to use the arguments from Section 4.2.2 to calculate the necessary ε-push-forwards modulo classes pulled back via ε. We obtain

$$\begin{aligned}&\sum_{d=0}^{\infty}\varepsilon_*\sum_{i=n'+1}^{n}\left(1+\sum_{\emptyset\neq T\subset\{1,\dots,d\}}\omega_i^{-1}\Delta_{iT}\right)\exp\left(\sum_{i=1}^{n'}s_ip_i\right)\\ &\qquad\sum_{j=0}^{\infty}(\zeta s)^{d-j}c_j(-\mathbb{B}_d)\frac{x^d}{(\zeta s)^{2d}d!}=\\ &\prod_{i=n'+1}^{n}\exp\left(\frac{u^\zeta}{\omega_i}-\log(\Phi')\left(\frac{\psi_i}{\zeta(v)s},\frac{x}{\zeta(v)s}\right)\right)\\ &\qquad\exp\left(\left\{\exp(-pD)\log(\Phi)\left(\frac{t}{\zeta(v)s},\frac{x}{\zeta(v)s}\right)\right\}_{\Delta}\right)\Big|_{t=1},\end{aligned}\tag{12}$$

where u^ζ is as in (10) and $p = p_1 + \cdots + p_{n'}$. Here we rather artificially introduced a variable t to make use of the bracket notation. Notice that in the first factor of this formula the factor corresponding to some $i \in \{n'+1, \dots, n\}$ depends only on ω_i, which dependends on the degree splitting of the chain, whereas the second factor is independent of the degree splittings.

Now we can again step back from the vertex and look at the global picture. With (12) we can give a new formula for $\nu_*^{vir}\varphi(\mathbf{a})$

$$\nu_*^{vir}\varphi(\mathbf{a}) = \left[\sum_{\Gamma,\zeta}\frac{1}{\mathrm{Aut}(\Gamma)}\xi_{\Gamma*}\left(\prod_v \mathrm{Vertex}_v^2\prod_e \mathrm{Edge}_e^2\prod_{i=1}^{n}\mathrm{Point}_i^2\right)\right]_{x^d\mathbf{p}^{\mathbf{a}}},$$

with new vertex, edge and point series. The term Vertex_v^2 already has the form as in the stable quotient relations

$$\begin{aligned}&\mathrm{Vertex}_v^2 = (\zeta(v)s)^{g(v)-1}\\ &\exp\left(-\frac{1}{2}\zeta(v)sp_{(v)}+\left\{\exp(p_{(v)}sD)\log(\Phi)\left(\frac{t}{\zeta(v)s},\frac{x}{(\zeta(v)s)^2}\right)\right\}_{\Delta^{(v)}}\Big|_{t=1}+V'^{\zeta(v)}\right),\end{aligned}$$

where

$$V'^{\zeta} = \sum_{j=0}^{\infty}(\zeta s)^{-j}c_j(\mathbb{E}^*) \in A^*(\overline{M}_{g(v),\mathbf{w}_v})[s,s^{-1}][\![x,s,s^{-1}]\!].$$

Furthermore the edge and point series are given by

$$\begin{aligned}\mathrm{Edge}^2{}_e =&\Phi'^{-1}\left(\frac{\psi_1}{\zeta_1 s},\frac{x}{\zeta_1 s}\right)\Phi'^{-1}\left(\frac{\psi_2}{\zeta_2 s},\frac{x}{\zeta_2 s}\right)\\ \sum_{d=1}^{\infty}x^d\sum_{\Gamma_e^d=(V_e,E_e)}&\frac{1}{\mathrm{Aut}(\Gamma_e^d)}\frac{\zeta_1 se^{u^{\zeta_1}/\omega_{\Gamma_e^d,1}}}{\omega_{\Gamma_e^d,1}-\psi_1}\frac{\zeta_2 se^{u^{\zeta_2}/\omega_{\Gamma_e^d,2}}}{\omega_{\Gamma_e^d,2}-\psi_2}\prod_{f\text{ edge}}\mathrm{Cont}(f)\prod_{v\text{ vertex}}\mathrm{Cont}(v)\end{aligned}$$

[8]To think of M_i as living on $\overline{M}_{g,(\mathbf{w},\varepsilon^{|i|})}$ one needs to choose a bijection $i \to \{1,\dots,|i|\}$ but the ε_i-push-forward is independent of that choice.

and

$$\begin{aligned}\mathrm{Point}_i^2 =& \Phi'^{-1}\left(\frac{\psi_i}{\zeta_i s}, \frac{x}{\zeta_i s}\right)\\ &\sum_{d=0}^{\infty} x^d \sum_{\Gamma_i^d=(V_i,E_i)} \frac{1}{\mathrm{Aut}(\Gamma_i^d)} \frac{\zeta_i s e^{u^{\zeta_i}/\omega_{\Gamma_i^d}}}{\omega_{\Gamma_i^d} - \psi_i} \prod_{f \text{ edge}} \mathrm{Cont}(f) \prod_{v \text{ vertex}} \mathrm{Cont}(v).\end{aligned}$$

Using Mumford's formula (pulled back from $\overline{M}_g$ to $\overline{M}_{g,\mathbf{w}}$) we find

$$\begin{aligned}V'^{\zeta} =& -\sum_{i\geq 1} \frac{B_{2i}}{2i\cdot(2i-1)} \kappa_{2i-1}(\zeta s)^{1-2i}\\ &-\sum_{i\geq 1} \frac{B_{2i}}{2i\cdot(2i-1)} \sum_{\Delta\in\mathcal{DG}} \frac{1}{\mathrm{Aut}(\Delta)} \xi_{\Delta,*}\left(\frac{\psi_a^{2i-1}+\psi_b^{2i-1}}{\psi_a+\psi_b}\right)(\zeta s)^{1-2i}.\end{aligned}$$

Notice that the edge series Edge_e^2 modulo a slight change in notation and the Φ'^{-1} factors is the series $E^{ij}(\psi_1,\psi_2,x)$ from Section 5, where i and j correspond to the color of the vertices e connects. Similarly we identify the point term Point_i^2 up to the Φ'^{-1}-factor with

$$2^{-a_i}\left(s^{a_i} P^{i0}(\psi_i, x) + (-s)^{a_i} P^{i\infty}(\psi_i, x)\right).$$

We finally obtain the relations by taking the s^c part of $\nu_*^{vir}\varphi(\mathbf{a})$ for $c<0$. So we replace everywhere s by t^{-1}, x by xt^2 and $p_{(v)}$ by $p_{(v)}t^{-1}$. After dividing out a common factor of t^e for

$$e = \sum_v (-g(v)+1) + 2d + |a| = -g + 1 + |E| + 2d + |a|$$

and introducing variables p_i for the a''_i we arrive at the stable quotient relations of Section 6.1.

6.3. **Evaluation of the relations.**

6.3.1. *Minor simplification.* With the same proof as in Section 4.3.2 the stable quotient relations are implied from the stable quotient relations in the case that the n'-tuple $\mathbf{a}'$ is only $\{0,1\}$-valued. The same holds trivially for $\mathbf{a}''$ because of the form of the point term.

Furthermore the relations in the case that a point is of weight 1 and a point is of weight slightly smaller than 1 are the same. This is because a point i of weight slightly smaller than 1 is not allowed to meet any other point, therefore the contribution of that point, which is solely in the vertex contribution, is

$$\exp\left(-\frac{1}{2}\zeta(p(i))p_i + p_i D\gamma(t\zeta(p(i))\psi_i, x)\right) \equiv 1 + p_i\zeta(p(i))\delta(t\zeta(p(i))\psi_i, x) \pmod{p_i^2}$$

This is the point term after suitably renaming p_i.

Therefore we can from now on treat points of weight 1 the same way as points of weight slightly less than 1.

6.3.2. *Edge terms.* The factor $\exp(V^\zeta)$ of the vertex contribution to the stable quotient relations contains intersections of classes supported on divisor classes of $\overline{M}_{g(v),\mathbf{w}_v}$. We want to reformulate the relations such that the vertex term only contains κ-, diagonal and ψ-classes corresponding to the markings. Some excess intersection calculations will be necessary to deal with $\exp(V^\zeta)$.

Proposition 4. *The set of stable quotient relations is equivalent to the following set of relations: Under the conditions of Proposition 3 it holds*

$$0 = \left[\sum_{\substack{\Gamma \in \mathcal{G} \\ \zeta : \Gamma \to \{\pm 1\}}} \frac{1}{\mathrm{Aut}(\Gamma)} \xi_{\Gamma *} \Big(\prod_v \mathrm{Vertex4}_v^{\zeta(v)} \prod_e \mathrm{Edge4}_e^{\zeta(v_1), \zeta(v_2)} \Big) \right]_{t^{r-|E|} x^d \mathbf{p}^{\mathbf{a}}},$$

with

$$\mathrm{Vertex4}_v^\zeta = \zeta^{g(v)-1} \exp\left(\frac{1}{2} \zeta p_{(v)} + \{\exp(p_{(v)} D) \gamma(t\zeta, x)\}_{\Delta^{(v)}} \right),$$

where $p_{(v)} = \sum_{i \in p^{-1}(v)} p_i$, *and*

$$t(\psi_1 + \psi_2) \mathrm{Edge4}_e^{\zeta_1, \zeta_2} = \frac{\zeta_1 + \zeta_2}{2} \exp(-\gamma'(t\zeta_1 \psi_1) - \gamma'(t\zeta_2 \psi_2)) + \zeta_1 \delta(t\zeta_1 \psi_1) + \zeta_2 \delta(t\zeta_2 \psi_2),$$

where γ' *is defined in the same way from* Φ' *as* γ *is from* Φ*:*

$$\gamma' = \sum_{i \geq 1} \frac{B_{2i}}{2i \cdot (2i-1)} t^{2i-1} + \log(\Phi')$$

Remark 3. As in Section 4.3.2 we can also write

$$\mathrm{Vertex4}_v^\zeta = \zeta^{g(v)-1} \exp\left(-\{\gamma\}^\zeta_{\kappa^{(v)}} + \sum_{i=1}^\infty \frac{\zeta^i}{i!} \{p_{(v)}^i D^{i-1} \delta\}^\zeta_{\Delta^{(v)}} \right).$$

The power ζ^i appears because of the t in $D = tx\frac{\mathrm{d}}{\mathrm{d}x}$.

The proof of Proposition 4 depends on the following lemma.

Lemma 7. *For a polynomial* f *in two variables we have*

$$\exp\left(\sum_{\Delta \in \mathcal{DG}} \frac{1}{|\mathrm{Aut}(\Delta)|} \xi_{\Delta,*}(f(\psi_a, \psi_b)) \right) =$$

$$\sum_{\Gamma \in \mathcal{G}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \xi_{\Gamma,*} \left(\prod_e \frac{\exp(-f(\psi_1^{(e)}, \psi_2^{(e)})(\psi_1^{(e)} + \psi_2^{(e)})) - 1}{-(\psi_1^{(e)} + \psi_2^{(e)})} \right),$$

where $\psi_i^{(e)}$ *are the two cotangent line classes belonging to edge* e.

Proof. Formally expanding the left hand side using the intersection formulas, for example described in [8, Appendix A], we can write it as a sum over stable graphs (Γ, E). Let us look at the term corresponding to a given graph Γ. By contracting all but one edge of Γ one obtains a divisor graph. This process gives a map $e_\Gamma : E \to \mathcal{DG}$. Counting the preimages of e_Γ gives a map $m_\Gamma : \mathcal{DG} \to \mathbb{N}_0$. In the formal expansion of the exponential on the left hand side each term also corresponds to a function $\sigma : \mathcal{DG} \to \mathbb{N}_0$.

A term contributes to a graph Γ if and only if $m_\Gamma \leq \sigma$, of the $|\sigma|$ intersections $|m_\Gamma| = |E|$ are transversal and the others are excess. In addition, a contributing term of the left hand side determines a partition $\mathbf{p}$ indexed by E of $\sigma = \sum_{e \in E} p_e$ such that $p_e(\Delta) = 0$ unless $e_\Gamma(e) = \Delta$.

With this we can explicitly write down $|\operatorname{Aut}(\Gamma)|$ times the Γ-contribution as

$$\begin{aligned}
&\xi_{\Gamma,*}\sum_{\sigma\geq m_\Gamma}\sum_{\mathbf{p}}\quad\prod_{\Delta\in\mathcal{DG}}\frac{1}{\sigma(\Delta)!}\binom{\sigma(\Delta)}{\mathbf{p}(\Delta)}\\
&\qquad\qquad\prod_e f(\psi_1^{(e)},\psi_2^{(e)})^{p_e(e_\Gamma(e))}(-(\psi_1^{(e)}+\psi_2^{(e)}))^{p_e(e_\Gamma(e))-1}\\
=&\xi_{\Gamma,*}\sum_{\sigma\geq m_\Gamma}\sum_{\mathbf{p}}\quad\prod_e\frac{1}{(p_e(e_\Gamma(e)))!}f(\psi_1^{(e)},\psi_2^{(e)})^{p_e(e_\Gamma(e))}(-(\psi_1^{(e)}+\psi_2^{(e)}))^{p_e(e_\Gamma(e))-1}\\
=&\xi_{\Gamma,*}\qquad\qquad\prod_e\sum_{i=1}^{\infty}\frac{1}{i!}f(\psi_1^{(e)},\psi_2^{(e)})^i(-(\psi_1^{(e)}+\psi_2^{(e)}))^{i-1}\\
=&\xi_{\Gamma,*}\qquad\qquad\prod_e\frac{\exp(-f(\psi_1^{(e)},\psi_2^{(e)})(\psi_1^{(e)}+\psi_2^{(e)}))-1}{-(\psi_1^{(e)}+\psi_2^{(e)})}.
\end{aligned}$$

Here the factor $(\sigma(\Delta)!)^{-1}$ comes from the exponential and

$$\prod_{\Delta\in\mathcal{DG}}\binom{\sigma(\Delta)}{\mathbf{p}(\Delta)}$$

comes from the choice of which intersections are excess.

Summing the contributions for all Γ finishes the proof. □

Proof of the proposition. We will apply the lemma in the case that

$$f(x_1,x_2)=-\sum_{i\geq 1}\frac{B_{2i}}{2i(2i-1)}\frac{x_1^{2i-1}+x_2^{2i-1}}{x_1+x_2},$$

but now we also need to take care of the coloring of the vertices.

For each graph $\Gamma\in\mathcal{G}$ with a coloring $\zeta:\Gamma\to\{\pm 1\}$ we can construct a new graph Γ_{red}, its *reduction*, by contracting all edges of Γ connecting two vertices of the same color. The induced coloring on Γ_{red} satisfies the property that neighboring vertices are differently colored. Let us call such a graph reduced. Having the same reduction also defines an equivalence relation on $\mathcal{G}$.

The idea is now to apply lemma 7 to each vertex of each graph Γ. In this way we get terms at each specialization Γ' of Γ in the same equivalence class of Γ.

Let us collect all the different contributions at a graph Γ' coming from graphs Γ. Recall the pull-back formula for the κ-classes

$$p_{v*}(\xi_\Gamma^*\kappa_i)=\kappa_i+\sum_e\psi_e^i,$$

where p_v denotes the projection map to the factor corresponding to each vertex v and the sum is over all outgoing edges at v. This implies that the contributions at Γ' all have the same vertex contribution up a sign and a factor

$$\exp(-\gamma'(t\zeta_1\psi_1^{(e)})-\gamma'(t\zeta_2\psi_2^{(e)}))$$

for each edge of Γ not in Γ' [9]. The edge terms corresponding to common edges do exactly coincide. The different factors split into a product over the connected components of the graph obtained by removing the edges which need to be contracted to obtain Γ'_{red}.

[9] γ' appears here instead of γ because $\kappa_{-1}=0$ while ψ^{-1} is not defined.

So let us look at just one connected component $\Gamma'' \subset \Gamma' \setminus \Gamma'_{red}$. We have to sum over the possibilities $E \subseteq E(\Gamma'')$ of contracting edges in Γ''. We thus have the contribution

$$\sum_{E(\Gamma'')=E\coprod F} \zeta^{|E|} \prod_{e\in F} \mathrm{Edge3}_e^{\zeta,\zeta}$$

$$\prod_{e\in E} \frac{\exp(-f(t\zeta\psi_1^{(e)}, t\zeta\psi_2^{(e)})(t\zeta\psi_1^{(e)}+t\zeta\psi_2^{(e)}))-1}{-(t\zeta\psi_1^{(e)}+t\zeta\psi_2^{(e)})} \exp(-\gamma'(t\zeta\psi_1^{(e)})-\gamma'(t\zeta\psi_2^{(e)}))$$

$$= \prod_{e\in E(\Gamma'')} \Big(\mathrm{Edge3}_e^{\zeta,\zeta} +$$

$$\zeta \frac{\exp(-f(t\zeta\psi_1^{(e)}, t\zeta\psi_2^{(e)})(t\zeta\psi_1^{(e)}+t\zeta\psi_2^{(e)}))-1}{-(t\zeta\psi_1^{(e)}+t\zeta\psi_2^{(e)})} \exp(-\gamma'(t\zeta\psi_1^{(e)})-\gamma'(t\zeta\psi_2^{(e)}))\Big)$$

$$= \prod_{e\in E(\Gamma'')} \mathrm{Edge4}_e^{\zeta,\zeta}.$$

Because of

$$\mathrm{Edge3}_e^{\zeta,-\zeta} = \mathrm{Edge4}_e^{\zeta,-\zeta}$$

we can replace Edge^3 by Edge^4 also for the edges connecting differently colored vertices. ☐

6.3.3. *Variable transformations.* Using the results of Section 4.3.3 we can give a new formulation of the stable quotient relations.

We have

$$0 = \left[\sum_{\substack{\Gamma\in\mathcal{G} \\ \zeta:\Gamma\to\{\pm1\}}} \frac{1}{\mathrm{Aut}(\Gamma)} \xi_{\Gamma*}\Big((1+4y)^{e_\Gamma} \prod_v \mathrm{Vertex5}_v^{\zeta(v)} \prod_e \mathrm{Edge5}_e^{\zeta(v_1),\zeta(v_2)}\Big)\right]_{u^{r-|E|}y^d\mathbf{p}^{\mathbf{a}}},$$

with

$$\mathrm{Vertex5}_v^{\zeta} = \zeta^{g(v)-1} \exp\left(-\left\{\sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}u^k y^j\right\}_{\kappa}^{\zeta} + \sum_{i=1}^{\infty}\frac{\zeta^i}{i!}\{p_{(v)}^i\delta_i\}_{\Delta}^{\zeta}\right),$$

$$u(\psi_1+\psi_2)\mathrm{Edge5}_e^{\zeta_1,\zeta_2} = \frac{\zeta_1+\zeta_2}{2}\exp\left(-\sum_{k=1}^{\infty}\sum_{j=0}^{k} c_{k,j}((u\zeta_1\psi_1)^k+(u\zeta_2\psi_2)^k)y^j\right.$$
$$\left. +\zeta_1\delta_1(u\zeta_1\psi_1)+\zeta_2\delta_1(u\zeta_2\psi_2)\right)$$

and the exponent

$$e_\Gamma = \frac{r+2d-2}{2} - \frac{\kappa_0}{4} - \frac{|\mathbf{a}|}{2} = \frac{r-g+2d-1-|\mathbf{a}|}{2}$$

under the condition of Proposition 3 on r.

We can assume that e_Γ is integral because otherwise the relation is zero since the term corresponding to a coloring ζ and the opposite coloring $-\zeta$ exactly cancel each other in this case.

Next we look as in Section 4.3.4 at the extremal coefficients of this series and obtain the FZ relations of Proposition 1.

6.4. **Final remarks.** Let $\mathcal{R}(g,\mathbf{w},r;S)$ denote the relation on $\overline{M}_{g,\mathbf{w}}$ of Proposition 1 in codimension r corresponding to $S\subset\{1,\dots,n\}$ viewed as a class in the formal strata algebra, i.e. the formal $\mathbb{Q}$-algebra generated by the symbols

$$\xi_{\Gamma*}\left(\prod_v M_v\right),$$

where Γ is a stable graph of $\overline{M}_{g,\mathbf{w}}$ and the M_v are formal monomials in κ-, ψ- and diagonal classes, modulo the relations given by the formal multiplication rules for boundary strata described in [8, Appendix A] and the relations between diagonal and ψ- classes from 4.2.1. One can describe formal analogs of the push-forwards and pull-backs along the forgetful, gluing and weight reduction maps.

By the way we have constructed the stable quotient relations, for $\mathbf{w}'\le\mathbf{w}$ the push-forward of $\mathcal{R}(g,\mathbf{w},r;S)$ via the weight reduction map is $\mathcal{R}(g,\mathbf{w}',r;S)$. Therefore the relations of Proposition 1 are (up to a constant factor) the push-forward of a subset of Pixton's generalized FZ relations.

As mentioned in the introduction more relations than in Proposition 1 can be obtained by taking for a partition σ with no part equal to 2 (mod 3) the class

$$\mathcal{R}(g,(\mathbf{w},1^{\ell(\sigma)}),r-|\lfloor\sigma/3\rfloor|;\bar{S})\prod_i\psi_{n+i}^{\lfloor\sigma_i/3\rfloor+1}$$

in $A^{r+\ell(\sigma)}(\overline{M}_{g,(\mathbf{w},1^{\ell(\sigma)})})$, where $\bar{S}$ equals S on the first n markings and is given by the remainders when dividing the parts of σ by 3 on the other markings, and pushing this class forward to $\overline{M}_{g,\mathbf{w}}$ under the forgetful map. For explicitly calculating this push-forward it is better to use the usual κ-classes $\tilde{\kappa}_i=\pi_*(c_1(\omega_\pi(D))^{i+1})$, which are related to the κ-classes we have used in this article by $\tilde{\kappa}_i=\kappa_i+\sum_{j=1}^n\psi_j^i$, in order to use Faber's formula for the push-forward of monomials in cotangent line classes [2]. Let us call these relations $\mathcal{R}(g,\mathbf{w},r;\sigma,S)$.

As in [21] even more generally one can look at the $\mathbb{Q}$-vector space $\mathcal{R}_{g,\mathbf{w}}$ generated by the relations obtained by choosing a boundary stratum corresponding to a dual graph Γ, taking a FZ relation $\mathcal{R}(g_i,\mathbf{w}_i,r;\sigma,S)M_i$ for any r, S, σ and monomial M_i in the diagonal and cotangent line classes on one of the components, arbitrary tautological classes on the other components and pushing this forward along ξ_Γ. Because of the compatibility with the birational weight reduction maps [21, Proposition 1] implies that the system $\mathcal{R}_{g,\mathbf{w}}$ of $\mathbb{Q}$-vector spaces cannot be tautologically enlarged, i.e. it is closed under formal push-forward and pull-back along forgetful and gluing maps as well as multiplication with arbitrary tautological classes.

As in [19] we have thrown away many of the stable quotient relations: We looked only at the extremal relations in Sections 4.3.4 and 6.3.3. However we should expect that these additional relations can also be expressed in terms of FZ relations.

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Paper B

Comparing tautological relations from the equivariant Gromov-Witten theory of projective spaces and spin structures

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Comparing tautological relations from the equivariant Gromov-Witten theory of projective spaces and spin structures

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Abstract

Pandharipande-Pixton-Zvonkine's proof of Pixton's generalized Faber-Zagier relations in the tautological ring of $\overline{M}_{g,n}$ has started the study of tautological relations from semisimple cohomological field theories. In this article we compare the relations obtained in the examples of the equivariant Gromov-Witten theory of projective spaces and of spin structures. We prove an equivalence between the $\mathbb{P}^1$- and 3-spin relations, and more generally between restricted $\mathbb{P}^m$-relations and similarly restricted $(m+2)$-spin relations. We also show that the general $\mathbb{P}^m$-relations imply the $(m+2)$-spin relations.

1 Introduction

The study of the Chow ring of the moduli space of curves was initiated Mumford in [11]. Because it is difficult to understand the whole Chow ring in general, the tautological subrings of classes reflecting the geometry of the objects parametrized by the moduli space were introduced. The tautological ring $R^*(\overline{M}_{g,n})$ is compactly described [2] as the smallest system

$$R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$$

of subrings compatible with push-forward under the tautological maps, i.e. the maps obtained from forgetting marked points or gluing curves along common markings.

There is a canonical set of generators parametrized by decorated graphs [5]. The formal vector space $\mathcal{S}_{g,n}$ generated by them, the *strata algebra*, therefore admits a surjective map to $R^*(\overline{M}_{g,n})$ and the structure of the tautological ring is determined by the kernel of this surjection. Elements of the kernel are called *tautological relations*.

In [15] A. Pixton proposed a set of (at the time conjectural) relations generalizing the relations of Faber-Zagier in $R^*(M_g)$. Furthermore, he conjectured that these give all tautological relations. The first proof [13] of the fact that the conjectural relations are actual relations (in cohomology) brought cohomological field theories (CohFTs) into the picture.

A CohFT on a free module V of finite rank over a base ring A is a system of classes $\Omega_{g,n}$ behaving nicely under pull-back via the tautological maps. A CohFT can also be used to give V the structure of a Frobenius algebra. The CohFT is called semisimple if, after possible base extension, the algebra V has a basis of orthogonal idempotents.

For semisimple CohFTs there is a conjecture by Givental [3] proven in some cases by himself and in full generality in cohomology by Teleman [16], giving a reconstruction of the CohFT from its genus 0, codimension 0 part and the data of a power series $R(z)$ of endomorphisms of V. The formula naturally lifts to the strata algebra.

To get relations from a semisimple cohomological field theory we can use that the reconstructed CohFT of elements in the strata algebra is in general only defined over an extension $B \leftarrow A$. However since we have started out with a CohFT over A, this implies that certain linear combinations of elements in the strata algebra have to vanish under the projection to the tautological ring.

This procedure was essentially used in the proof [13] in the special example of the CohFT defined from Witten's 3-spin class. There the base ring is a polynomial ring in one variable but the reconstructed CohFT seems to have poles in this variable.

In [14] (in preparation) the authors construct tautological relations using Witten's r-spin class for any $r \geq 3$. Given a list of integers $a_1, \dots, a_n \in \{0, \dots, r-2\}$, Witten's class $W_{g,n}(a_1, \dots, a_n)$ is a cohomology class on $\overline{M}_{g,n}$ of pure degree

$$D_{g,n}(a_1, \dots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^{n} a_i}{r}.$$

Witten's class can be "shifted" by any vector in the vector space $\langle e_0, \dots, e_{r-2} \rangle$ to obtain a semisimple CohFT. In practice, the authors use two particular shifts for which the answer can be explicitly computed. Shifted Witten's class is of mixed degree: more precisely, the degrees of its components go from 0 to $D_{g,n}(a_1, \dots, a_n)$. On the other hand, the Givental-Teleman classificiation of semisimple CohFTs gives an expression of the shifted Witten class in terms of tautological classes. The authors conclude that the components of this expression beyond degree $D_{g,n}(a_1, \dots, a_n)$ are tautological relations.

This article studies how relations from spin structures are related to the relations obtained from the CohFT defined from the equivariant Gromov-Witten theory of projective spaces. The following two theorems are our main results.

Theorem 1 (rough version)**.** *The relations obtained from the equivariant Gromov-Witten theory of $\mathbb{P}^m$ imply the $(m+2)$-spin relations.*

Theorem 2 (rough version)**.** *A special restricted set of relations from equivariant $\mathbb{P}^m$ is equivalent to a corresponding restricted set of $(m+2)$-spin relations. For $\mathbb{P}^1$ and 3-spin no restriction is necessary.*

Since for equivariant $\mathbb{P}^m$ the reconstruction holds in Chow, Theorem 1 implies that the higher spin relations also hold in Chow.

We will give strong evidence that the method of proof for Theorem 2 cannot directly be extended to an equivalence between the full $\mathbb{P}^m$- and

$(m+2)$-spin relations for $m > 2$. Possibly, there are more $\mathbb{P}^m$- than $(m+2)$-spin relations.

Any of the theorems gives another proof of the fact that Pixton's relations hold in Chow. In fact, the proof of Theorem 1 in the case $m = 1$ is essentially a simplified version of the author's previous proof in [8].

This article does not give a comparison between relations from CohFTs of different dimensions, nor does it consider all relations from equivariant $\mathbb{P}^m$. On the other hand, if indeed Pixton's relations are all tautological relations, the 3-spin relations have to imply the relations from any other semisimple CohFT. Yet, for example it not clear how the 4-spin relations can be written in terms of 3-spin relations.

The article is structured as follows. In Section 2 we give definitions of CohFTs, discuss the R-matrix action on CohFTs and the reconstruction result. We then in Section 2.5 turn to the two examples of equivariant $\mathbb{P}^m$ and the CohFT from the A_{m+1}-singularity. In Section 2.6 we describe the general procedure of obtaining relations from semisimple CohFTs and general methods of proving that the relations from one CohFT imply the relations from another. We then state precise versions of Theorem 1 and 2. Section 3 discusses explicit expression of the R-matrices in both theories in terms of asymptotics of oscillating integrals. The constraints following from these expressions will be used in the next sections. We also note a connection to Airy functions. Section 4 and Section 5 give proofs of Theorem 1 and 2. Finally, Section 6 gives evidence why, with the methods used in the proofs of the theorems, an equivalence between $\mathbb{P}^m$- and $(m+2)$-spin relations cannot be established. Since the reconstruction result of Givental we use to get relations in Chow has never appeared explicitly in the literature, we recall its proof in Appendix A.

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2 Cohomological field theories

2.1 Definitions

Cohomological field theories were first introduced by Kontsevich and Manin in [10] to formalize the structure of classes from GW-theory. Let A be an integral, commutative $\mathbb{Q}$-algebra, V a free A-module of finite rank and η a non-singular bilinear form on V.

Definition 1. A cohomological field theory (CohFT) Ω on (V, η) is a system

$$\Omega_{g,n} \in A^*(\overline{M}_{g,n}) \otimes_{\mathbb{Q}} (V^*)^{\otimes n}$$

of multilinear forms with values in the Chow ring of $\overline{M}_{g,n}$ satisfying the following properties:

Symmetry $\Omega_{g,n}$ is symmetric in its n arguments

Gluing The pull-back of $\Omega_{g,n}$ via the gluing map

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$$

is given by the direct product of Ω_{g_1,n_2+1} and Ω_{g_2,n_2+1} with the bivector η^{-1} inserted at the two gluing points. Similarly for the gluing map $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ the pull-back of $\Omega_{g,n}$ is given by $\Omega_{g-1,n+2}$ with η^{-1} inserted at the two gluing points.

Unit There is a special element $\mathbf{1} \in V$ called the *unit* such that

$$\Omega_{g,n+1}(v_1, \ldots, v_n, \mathbf{1})$$

is the pull-back of $\Omega_{g,n}(v_1, \ldots, v_n)$ under the forgetful map and

$$\Omega_{0,3}(v, w, \mathbf{1}) = \eta(v, w).$$

Definition 2. The quantum product $(u, v) \mapsto uv$ on V with unit $\mathbf{1}$ is defined by the condition

$$\eta(uv, w) = \Omega_{0,3}(u, v, w). \tag{1}$$

Definition 3. A CohFT is called semisimple if there is a base extension $A \to B$ such that the algebra $V \otimes_A B$ is semisimple.

2.2 First Examples

Example 1. For each Frobenius algebra there is the *trivial CohFT* (also called *topological field theory* or TQFT) $\Omega_{g,n}$ characterized by (1) and that

$$\Omega_{g,n} \in A^0(\overline{M}_{g,n}) \otimes (V^*)^{\otimes n}.$$

Let us record an explicit formula for Appendix A: In the case that the Frobenius algebra is semisimple, there is a basis ϵ_i of orthogonal idempotents of V and

$$\tilde{\epsilon}_i = \frac{\epsilon_i}{\sqrt{\Delta_i}},$$

where $\Delta_i^{-1} = \eta(\epsilon_i, \epsilon_i)$, is a corresponding orthonormal basis of *normalized idempotents*. We have

$$\Omega_{g,n}(\tilde{\epsilon}_{i_1}, \ldots, \tilde{\epsilon}_{i_n}) = \begin{cases} \sum_j \Delta_{i_j}^{g-1}, & \text{if } n = 0, \\ \Delta_{i_1}^{\frac{2g-2+n}{2}}, & \text{if } i_1 = \cdots = i_n, \\ 0, & \text{else.} \end{cases}$$

Example 2. The Chern polynomial $c_t(\mathbb{E})$ of the Hodge bundle $\mathbb{E}$ gives a 1-dimensional CohFT over $\mathbb{Q}[t]$.

Example 3. Let X be a smooth, projective variety such that the cycle class map gives an isomorphism between Chow and cohomology rings. Let $A = \mathbb{Q}[\![q^\beta]\!]$ be its Novikov ring. Then the Gromov-Witten theory of X defines a CohFT based on the A-module $A^*(X) \otimes A$ by the definition

$$\Omega_{g,n}(v_1, \ldots, v_n) = \sum_\beta \pi_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(v_i) \cap [\overline{M}_{g,n}(X,\beta)]^{vir} \right) q^\beta,$$

where the sum ranges over effective, integral curve classes, ev_i is the i-th evaluation map and π is the forgetful map $\pi : \overline{M}_{g,n}(X,\beta) \to \overline{M}_{g,n}$. The gluing property follows from the splitting axiom of virtual fundamental classes. The fundamental class of X is the unit of the CohFT and the unit axioms follow from the identity axiom in GW-theory.

For a torus action on X, this example can be enhanced to give a CohFT from the equivariant GW-theory of X.

2.3 The R-matrix action

Definition 4. The (upper part of the) symplectic loop group is defined as the subgroup of the group of endomorphism valued power series $R = 1 + O(z)$ in z satisfying the symplectic condition

$$\eta(R(z)v, R(-z)w) = \eta(v, w)$$

for all vectors v and w.

An action of this group on the space of CohFTs makes it interesting for us. In its definition the endomorphism valued power series R is evaluated at cotangent line classes and applied to vectors.

Given a CohFT $\Omega_{g,n}$ the new CohFT $R\Omega_{g,n}$ takes the form of a sum over dual graphs Γ

$$R\Omega_{g,n}(v_1, \ldots, v_n) = \sum_\Gamma \frac{1}{\mathrm{Aut}(\Gamma)} \xi_* \left(\prod_v \sum_{k=0}^\infty \frac{1}{k!} \varepsilon_* \Omega_{g_v, n_v+k}(\ldots) \right),$$

where $\xi : \prod_v \overline{M}_{g_v,n_v} \to \overline{M}_{g,n}$ is the gluing map of curves of topological type Γ from their irreducible components, $\varepsilon : \overline{M}_{g_v,n_v+k} \to \overline{M}_{g_v,n_v}$ forgets the last k markings and we still need to specify what is put into the arguments of $\prod_v \Omega_{g_v,n_v+k_v}$.

- Into each argument corresponding to a marking of the curve, put $R^{-1}(\psi)$ applied to the corresponding vector.
- Into each pair of arguments corresponding to an edge put the bivector

  $$\frac{R^{-1}(\psi_1)\eta^{-1}R^{-1}(\psi_2)^t - \eta^{-1}}{-\psi_1 - \psi_2} \in \mathrm{Hom}(V^*, V)[\![\psi_1, \psi_2]\!] \cong V^{\otimes 2}[\![\psi_1, \psi_2]\!],$$

  where one has to substitute the ψ-classes at each side of the normalization of the node for ψ_1 and ψ_2. By the symplectic condition this is well-defined.

- Into each of the additional arguments for each vertex put

$$T(\psi) := \psi(1 - R^{-1}(\psi))\mathbf{1},$$

where ψ is the cotangent line class corresponding to that vertex. Since $T(z) = O(z^2)$ the above k-sum is finite.

Reconstruction Conjecture (Givental)**.** *The R-matrix action is free and transitive on the space of semisimple CohFTs based on a given Frobenius algebra.*

Theorem 3 (Givental[3])**.** *Reconstruction for the equivariant GW-theory of toric targets holds in Chow.*

Theorem 4 (Teleman[16])**.** *Reconstruction holds in cohomology.*

Remark 1. Givental's original conjecture was only stated in terms of the descendent integrals of the CohFT and there is no explicit proof of Theorem 3 in the literature. Therefore in Appendix A we recall the well-known lift of Givental's proof to CohFTs.

Example 4. By Mumford's Grothendieck-Riemann-Roch calculation [11] the single entry of the R-matrix taking the trivial one-dimensional CohFT to the CohFT from Example 2 is given by

$$\exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)}(tz)^{2i-1}\right),$$

where B_{2i} are the Bernoulli numbers, defined by

$$\sum_{i=0}^{\infty} B_i \frac{x^i}{i!} = \frac{x}{e^x - 1}.$$

More generally, if we consider a more general CohFT given by a product of Chern polynomials (in different variables) of the Hodge bundle, the R-matrix from the trivial CohFT is the product of the R-matrices of the factors.

2.4 Frobenius manifolds and the quantum differential equation

There is a natural way to deform a CohFT $\Omega_{g,n}$ on V over A to a CohFT over $A[\![V]\!]$. For a basis $\{e_\mu\}$ of V let

$$p = \sum t^\mu e_\mu$$

be a formal point on V. Then the deformed CohFT is given by

$$\Omega^p_{g,n}(v_1, \dots, v_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Omega_{g,n+k}(v_1, \dots, v_n, p, \dots, p).$$

Notice that the deformation is constant in the direction of the unit.

The quantum product on the deformed CohFT gives V the structure of a (formal) *Frobenius manifold* [1]. The e_μ induce flat vector fields on

V corresponding to the *flat coordinates* t^μ. Greek indices will stand for flat coordinates with an exception stated in Section 2.5.

A Frobenius manifold is called *conformal* if it admits an *Euler vector field*, i.e. a vector field E of the form

$$E = \sum_\mu (\alpha_\mu t^\mu + \beta_\mu) \frac{\partial}{\partial t^\mu},$$

such that the quantum product, the unit and the metric are eigenfunctions of the Lie derivative L_E with eigenvalues 1, -1 and $2-\delta$ respectively. Here δ is a rational number called conformal dimension. Assuming that A itself is the ring of (formal) functions of a variety X we say that the Frobenius manifold is *quasi-conformal* if there is vector field E on $X \times V$ satisfying the axioms of an Euler vector field.

A CohFT $\Omega_{g,n}$ is called *homogeneous* (*quasi-homogeneous*) if its Frobenius manifold is conformal (quasi-conformal) and the extended CohFT is an eigenvector of of L_E of eigenvalue $(g-1)\delta + n$. As the name suggests CohFTs are homogeneous if they carry a grading such that all natural structures are homogeneous with respect to the grading.

We say that the Frobenius manifold V is *semisimple* if there is a basis of idempotent vector fields ϵ_i defined after possible base extension of A. The idempotents can be formally integrated to *canonical coordinates* u_i. We will use roman indices for them. Let $\mathbf{u}$ be the diagonal matrix with entries u_i and Ψ be the transition matrix from the basis of normalized idempotents corresponding to the u_i to the flat basis e_i.

The R-matrix from the trivial theory to Ω^p satisfies a differential equation which is related to the *quantum differential equation*

$$z \frac{\partial}{\partial t^\alpha} S_j = e_\alpha \star S_j$$

for vectors S_j. We assemble the S_j into a matrix S.

Proposition 1 (see [4])**.** *If V is semisimple and after a choice of canonical coordinates u_i has been made, there exists a fundamental solution S to the quantum differential equation of the form*

$$S = \Psi R e^{\mathbf{u}/z}, \tag{2}$$

such that R satisfies the symplectic condition $R(z)R^t(-z) = 1$. The matrix R is unique up to right multiplication by a diagonal matrix of the form

$$\exp(a_1 z + a_3 z^3 + a_5 z^5 + \cdots)$$

for constant diagonal matrices a_i.

In the case that there exists an Euler vector field E, there is a unique matrix R defined from a fundamental solution S by (2) *which satisfies the homogeneity*

$$z \frac{\mathrm{d}}{\mathrm{d}z} R + L_E R = 0.$$

Such an R automatically satisfies the symplectic condition.

Remark 2. The matrix R should be thought as the matrix representation of an endomorphism in the basis of normalized idempotents. The symplectic condition in Proposition 1 is then the same as in Definition 4.

Remark 3. The exponential in (2) has to be thought as a formal expression. All the quantities in Proposition 1 are only defined after base change of A necessary to define the canonical coordinates.

Remark 4. The quantum differential equation is equivalent to the differential equation

$$[R, \mathrm{d}\mathbf{u}] + z\Psi^{-1}\mathrm{d}(\Psi R) = 0 \tag{3}$$

for R.

In the conformal case Teleman showed that the uniquely determined homogeneous R-matrix of Proposition 1 is the one appearing in the reconstruction, taking the trivial theory to the given one.

Equivariant projective spaces $\mathbb{P}^m$ only give a quasi-conformal Frobenius manifold. However Givental showed, and we will recall the proof in Appendix A, that in this case in the reconstruction one should take R such that in the classical limit $q \to 0$ it assumes the diagonal form

$$R|_{q=0} = \exp(\mathrm{diag}(b_0, \ldots, b_m)), \tag{4}$$

where, using the notation from Section 2.5,

$$b_j = \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} \sum_{l \neq j} \left(\frac{z}{\lambda_l - \lambda_j} \right)^{2i-1}.$$

The R-matrix is uniquely determined by this additional property and the homogeneity.

2.5 The two CohFTs

The cohomological field theory corresponding to the A_{m+1}-singularity $f(X) = X^{m+2}/(m+2)$ is defined using Witten's $(m+2)$-spin class on the moduli of curves with $(m+2)$-spin structures. See [13] for a discussion of different constructions of Witten's class. In comparison to [13] we use a different normalization for Witten's class and a different basis for the free module in order to have a more direct comparison to the $\mathbb{P}^m$-theory.

The CohFT is based on the rank $(m+1)$ free module of versal deformations

$$f_t(X) = \frac{X^{m+2}}{m+2} + t^m X^m + \cdots + t^1 X + t^0$$

of f. In this article, using the deformation from Section 2.4, we will view the CohFT as being based on

$$k_{A_{m+1}} = \mathbb{Q}[t^1, \ldots, t^m],$$

the space of regular functions on the Frobenius manifold where the t^0-coordinate vanishes. Because of dimension constraints we do not need to look at formal functions, and because the CohFT stays constant along the t^0 direction we can restrict to the $(t^0 = 0)$-subspace.

The algebra structure is given by $k_{A_{m+1}}[X]/(f_t')$, where X^μ corresponds to $\frac{\partial}{\partial t^\mu}$. The metric is given by the residue pairing

$$\eta(a, b) = \frac{1}{2\pi \imath} \oint \frac{ab}{f_t'(X)} \mathrm{d}X.$$

Written as a matrix in the basis $1, \dots, X^m$, the metric η has therefore zeros above the antidiagonal, ones at the antidiagonal and again zeros in the first antidiagonal below it. Notice also that η has no dependence on t^1. Therefore, while the t^μ do not give a basis of flat vector fields on the Frobenius manifold, there is a triangular matrix independent of t^1, sending the $1, \dots, X^m$ to a basis of flat vector fields such that X is mapped to itself. With this we can pretend that the t^μ were flat coordinates if we consider in the quantum differential equation only differentiation by t^1.

For $(\mathbb{C}^*)^{m+1}$-equivariant $\mathbb{P}^m$ the CohFT is based on the equivariant Chow ring

$$A^*_{(\mathbb{C}^*)^{m+1}}(\mathbb{P}^m)[\![q]\!] \cong k_{\mathbb{P}^m}[H]/\prod_{i=0}^{m}(H-\lambda_i),$$

of $\mathbb{P}^m$, an $(m+1)$-dimensional free module over

$$k_{\mathbb{P}^m} = \mathbb{Q}[\lambda_0, \dots, \lambda_m][\![q]\!],$$

and depends on the Novikov variable q and the torus parameters λ_i. We will not consider the deformation from Section 2.4. The algebra structure is given by the small quantum equivariant Chow ring

$$QA^*_{(\mathbb{C}^*)^{m+1}}(\mathbb{P}^m) \cong k_{\mathbb{P}^m}[H]/\left(\prod_{i=0}^{m}(H-\lambda_i) - q\right)$$

and the pairing is the Poincaré pairing

$$\eta(a,b) = \frac{1}{2\pi\imath}\oint \frac{ab}{\prod_{i=0}^{m}(H-\lambda_i)}\mathrm{d}H$$

in the equivariant Chow ring.

To match up this data we set

$$\begin{aligned} X &= H - \bar{\lambda}, \\ X^{m+1} + \sum_{\mu=0}^{m-1}(\mu+1)t^{\mu+1}X^\mu &= \prod_{i=0}^{m}(X + \bar{\lambda} - \lambda_i) - q, \end{aligned}$$

where

$$\bar{\lambda} = \sum_{i=0}^{m}\frac{\lambda_i}{m+1}.$$

So in particular

$$t^1 = -q + \prod_{i=0}^{m}(\bar{\lambda} - \lambda_i) =: -q - \lambda$$

and we have described a ring map

$$\Phi : k_{A_{m+1}}[\lambda] \to k_{\mathbb{P}^m},$$

whose image are the polynomials, symmetric in the torus parameters and vanishing if all torus parameters coincide. Therefore, after base extension, the Frobenius algebras from the A_{m+1}-singularity and equivariant $\mathbb{P}^m$ match completely up.

On the $\mathbb{P}^m$-side, let Q_i be the power series solution to

$$\prod_{i=0}^{m}(Y+\bar{\lambda}-\lambda_i)=q$$

with limit $\lambda_i-\bar{\lambda}$ as $q\to 0$. In particular, the Q_i are solutions to

$$Y^{m+1}+\sum_{\mu=0}^{m-1}(\mu+1)t^{\mu+1}Y^{\mu}.$$

On the A_{m+1}-side, let the Q_i be the solutions to this equation in any order. On both sides we can then define

$$\Delta_i=\prod_{j\neq i}(Q_i-Q_j)=(m+1)Q_i^m-\sum_{\mu=1}^{m-1}(\mu+1)\mu t^{\mu+1}Q_i^{\mu-1}$$

and the *discriminant*

$$\mathrm{disc}=\prod_i\Delta_i\in k_{A_{m+1}}.$$

The choice of the Q_i gives a bijection between the idempotents

$$\epsilon_i=\frac{\prod_{j\neq i}(X-Q_j)}{\Delta_i}.$$

We will also need to make a choice of square roots of the Δ_i to be able to define the normalized idempotents

$$\tilde{\epsilon}_i=\frac{\prod_{j\neq i}(X-Q_j)}{\sqrt{\Delta_i}}.$$

The A_{m+1}-theory is conformal with Euler vector field

$$E=\sum_{i=1}^{m}\frac{m+2-i}{m+2}t^{\mu}\frac{\partial}{\partial t^{\mu}},$$

while the equivariant $\mathbb{P}^m$-theory is semi-conformal with Euler vector field

$$E=(m+1)q\frac{\partial}{\partial q}+\sum_{i=0}^{m}\lambda_i\frac{\partial}{\partial\lambda_i}.$$

2.6 Relations from CohFTs

Let Ω be a semisimple CohFT defined on V over A. Formal properties of the reconstruction theorem will imply tautological relations. The main point is that the R-matrix from the trivial theory written in flat coordinates lives only in

$$\mathrm{End}(V\otimes_A B)[\![z]\!],$$

for some $\mathbb{Q}$-algebra extension B[1] of A. Let C be the A-module quotient fitting into the exact sequence

$$0\to A\to B\xrightarrow{p}C\to 0. \tag{5}$$

[1]In our examples $B=A[\mathrm{disc}^{-1}]$.

The reconstruction gives elements

$$\overline{\Omega}_{g,n} \in \mathcal{S}_{g,n} \otimes (V^*)^{\otimes n} \otimes B.$$

However since we have started out with a CohFT defined over A, we know that the projection of

$$p(\overline{\Omega}_{g,n}) \in \mathcal{S}_{g,n} \otimes (V^*)^{\otimes n} \otimes C$$

to $R^*(\overline{M}_{g,n}) \otimes (V^*)^{\otimes n} \otimes C$ has to vanish[2]. Since C is a $\mathbb{Q}$-vector space, we obtain a system of vector spaces $T^{\Omega}_{g,n}$ of relations. The complete system $\bar{T}^{\Omega}_{g,n}$ of *tautological relations obtained from the CohFT* Ω is the vector space generated by

$$\xi_*(\pi_*(T^{\Omega}_{g_1,n_1+m}P) \times \mathcal{S}_{g_2,n_2} \times \cdots \times \mathcal{S}_{g_k,n_k}),$$

where P is the vector space of polynomials in ψ-classes, and ξ_* and π_* are the formal analogues of the push-forwards along gluing and forgetful maps.

We say that a vector space of tautological relations $T_{g,n}$ implies another $T'_{g,n}$ if the vector space, obtained from $T_{g,n}$ by the completion process as described right above, is contained in $T'_{g,n}$. Using this definition we can also define an equivalence relation between vector spaces of tautological relations.

Let us describe two relation preserving actions on the space of all CohFTs on V over A. The first is an action of the multiplicative monoid of A. The action of $\varphi \in A$ is given by multiplication by φ^d in codimension d. This replaces the R-matrix $R(z)$ of the theory by $R(\varphi z)$. Since multiplication by φ is well-defined in C, relations are preserved. The second action is the action of an R-matrix defined over A.

The second action automatically proves equivalence of relations since R-matrices are always invertible. Similarly, the first action proves equivalence if φ is invertible.

Extending scalars also preserves relations. By this we mean tensoring Ω with $A \to A'$ under the condition that this preserves the exactness of the sequence (5). We call the special case when $A' = A/I$ for some ideal I of A a *limit*. If $C \to C \otimes_A A'$ is injective, extending scalars proves an equivalence of relations.

Let us again state our now well-defined results.

Theorem 1. *The relations from the equivariant Gromov-Witten theory of* $\mathbb{P}^m$ *imply the* $(m+2)$*-spin relations, both CohFTs as defined in Section 2.5.*

The main statement necessary to be proven here is that the R-matrix for $\mathbb{P}^m$ after replacing $z \mapsto z\lambda^{-1}$ admits the limit $\lambda^{-1} \to 0$ and that this limit is the R-matrix for the A_{m+1}-theory. In order for this to make sense, one uses the matchup from Section 2.5 and views both as being defined over

$$\mathbb{Q}[\![\lambda_0, \ldots, \lambda_m, q]\!][\lambda^{-1}]$$

In Section 3 we will see that for both original theories to define the R-matrix it is enough to localize by disc. So the extension of scalars does not lose relations.

[2]Assuming that reconstruction holds in this case.

Motivated from Section 3.1 let us call the limit $t^2, \dots, t^m = 0$ the *Airy limit*. For $\mathbb{P}^m$ the Airy limit concretely means, assuming the sum of all torus weights is zero, that we restrict ourselves to the case that up to a factor the torus weights are the $(m+1)$-th roots of unity.

Theorem 2. *In the Airy limit the $\mathbb{P}^m$- and $(m+2)$-spin relations are equivalent.*

The main point for the proof is to show there is a series

$$\varphi \in \lambda\mathbb{Q}[\![t^1\lambda^{-1}]\!],$$

and an R-matrix R without poles in disc such that the Airy limit $\mathbb{P}^m$-R-matrix is obtained from the Airy limit A_{m+1}-R-matrix by applying the transformation $z \mapsto z\varphi$, followed by the action of R. We will show in the proof that there is only one possible choice for φ. For Theorem 2 both theories can be viewed as living over

$$\mathbb{Q}[\![\lambda_0, \dots, \lambda_m, q, t^1\lambda^{-1}]\!][\lambda^{-1}]/(t^1 + q + \lambda, t^2, \dots, t^m).$$

In Section 6 we will give evidence that the method of proof of Theorem 2 does not work outside the Airy limit. What we will show is that assuming a procedure as in the proof of Theorem 2 exists and is well-defined in the Airy limit, the information that φ was unique in the limit implies that the R-matrix in the R-matrix action cannot be defined over the base ring.

Relations from degree vanishing

The more classical way of [13] and [14] to obtain tautological relations works by considering cohomological degrees: Assume that Ω is in addition quasi-homogenous for an Euler vector field E and that all β_i vanish and all α_i are positive. Then the quasi-homogeneity implies that the cohomological degree of $\Omega_{g,n}(\frac{\partial}{\partial t^{i_1}}, \dots, \frac{\partial}{\partial t^{i_n}})$ is bounded by

$$(g-1)\delta + n - \sum_j \alpha_{i_j}.$$

However the reconstructed theory might also contain terms of higher cohomological degree. These thus have to vanish, giving tautological relations.

Notice that these relations coming from degree considerations are implied from the relations we have described previously: With respect to the grading on B induced by the Euler vector field, no element of A has negative degree. Therefore the negative degree parts of B and C are isomorphic. Thus, the homogeneity of the CohFT implies that the degree vanishing relations are obtained from the previous relations by restricting to the negative degree part of C.

The way of obtaining tautological relations by looking at poles in the discriminant has already previously been studied by D. Zvonkine.

3 Oscillating integrals

3.1 For the A_{m+1}-singularity

We want to describe the A_{m+1}-R-matrix in terms of asymptotics of oscillating integrals. For the purposes of this article the integrals can be treated as purely formal objects.

The quantum differential equation with one index lowered says that

$$\begin{aligned} z\frac{\partial}{\partial t^1} S_{\mu k} =& S_{(\mu+1)k}, && \text{for } \mu < m, \\ z\frac{\partial}{\partial t^1} S_{mk} =& -\sum_{\mu=0}^{m-1} (\mu+1) t^{\mu+1} S_{\mu k}, \end{aligned}$$

where the Greek indices stand for components in the basis of the X^μ. It is not difficult to see that the oscillating integrals

$$\frac{1}{\sqrt{-2\pi z}} \int_{\Gamma_k} x^\mu \exp(f_t(x)/z) \mathrm{d}x,$$

where f_t as before is the deformed singularity, for varying cycles Γ_k, if convergent, provide solutions to this system of differential equations, and also satisfy homogeneity with respect to the Euler vector field.

For generic choices of parameters, to each critical point Q_k there corresponds a cycle Γ_k constructed via the Morse theory of $\Re(f_t(x)/z)$, which moves through that critical point in the direction of steepest descent and avoids all other critical points. By moving to the critical point and scaling coordinates we obtain

$$S_{\mu k} = \frac{e^{u_k/z}}{\sqrt{2\pi\Delta_k}} \int \left(\frac{x(-z)^{1/2}}{\sqrt{\Delta_k}} + Q_k \right)^\mu \\ \exp\left(-\sum_{l=2}^{m+2} \frac{x^l (-z)^{(l-2)/2}}{l!} \frac{f_t^{(l)}(Q_k)}{\Delta_k^{l/2}} \right) \mathrm{d}x,$$

where $u_k = f_t(Q_k)$. By the method of steepest descend, we obtain the asymptotics as $z \to 0$ by expanding the integrand as a formal power series in z and integrating from $-\infty$ to ∞. Since the $(l = 2)$-term in the sum is $-x^2/2$, we can use the formula for the moments of the Gaussian distribution to write the asymptotics of $\sqrt{\Delta_k} e^{-u_k/z} S_{\mu k}$ as a formal power series in z with values in $k_{A_{m+1}}[Q_k, \Delta_k^{-1}]$.

The entries of the R-matrix are then given by

$$R_{ik} \asymp \frac{1}{\sqrt{\Delta_i}} e^{-u_k/z} \prod_{j\neq i} \left(\frac{\partial}{\partial t^1} - Q_j \right) S_{0k}.$$

Noticing that the change of basis from normalized idempotents to the basis $1, X, \ldots, X^m$ can be defined over $k_{A_{m+1}}[Q_k, \Delta_k^{-1/2}]$, recalling that $\mathrm{disc} = \prod \Delta_i$ and applying Galois theory, we see that the endomorphism R is defined over $k_{A_{m+1}}[\mathrm{disc}^{-1}]$.

In the Airy limit $t^2, \ldots, t^m \to 0$ the quantum differential equation becomes the slightly modified higher Airy differential equation [9]

$$\left(z\frac{\partial}{\partial t^1}\right)^{m+1} S_{0k} = -t^1 S_{0k}.$$

The entries of the R-matrix in this case are therefore related to the asymptotic expansions of the higher Airy functions and their derivatives when their complex argument approaches ∞.

In the case of the A_2-singularity we do not need to take any limit and discover the hypergeometric series A and B of Faber-Zagier in the expansions of the (slightly modified) usual Airy function

$$\frac{e^{\frac{2}{3}(t^1)^{3/2}/z}}{\sqrt{-2\pi z}} \int_{\Gamma_k} e^{\left(\frac{x^3}{3}+t^1 x\right)/z} \mathrm{d}x \asymp \frac{1}{\sqrt{2\pi\Delta}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} - \frac{x^3}{3}\frac{\sqrt{-z}}{\Delta^{3/2}}\right) \mathrm{d}x$$

$$\asymp \frac{1}{\sqrt{\Delta}} \sum_{i=0}^{\infty} \frac{(6i-1)!!}{(2i)!} \left(\frac{-z}{9\Delta^3}\right)^i = \frac{1}{\sqrt{\Delta}} \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{-z}{72\Delta^3}\right)^i$$

and a derivative of it

$$\frac{e^{\frac{2}{3}(t^1)^{3/2}/z}}{\sqrt{-2\pi z}} \int_{\Gamma_k} x e^{\left(\frac{x^3}{3}+t^1 x\right)/z} \mathrm{d}x \asymp \frac{\sqrt{-t^1}}{\sqrt{\Delta}} \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \frac{1+6i}{1-6i} \left(\frac{-z}{72\Delta^3}\right)^i.$$

Here $\Delta = 2\sqrt{-t^1}$. The cycle Γ_k determines which square-root of $(-t^1)$ we take.

3.2 For equivariant $\mathbb{P}^m$

Givental [3] has given explicit solutions to the quantum differential equation for projective spaces in the form of complex oscillating integrals. Let us recall their definition and see how they behave in the match up with the $(m+2)$-spin theory.

Using the divisor axiom of Gromov-Witten invariants, the quantum differential equation implies the differential equations

$$(D + \lambda_i)S_i = H \star S_i.$$

for the fundamental solutions S_i at the origin. Here we have written $D = zq\frac{\partial}{\partial q}$. Equivalently, the equation says

$$D(S_i e^{\ln(q)\lambda_i/z}) = H \star S_i e^{\ln(q)\lambda_i/z}.$$

Therefore the entries of S with one index lowered satisfy

$$(D - \bar{\lambda})(S_{\mu i} e^{\ln(q)\lambda_i/z}) = S_{(\mu+1)i} e^{\ln(q)\lambda_i/z},$$

where $S_{(m+1)i}$ is defined such that

$$\prod_{j=0}^{m}(D - \lambda_j)(S_{0i} e^{\ln(q)\lambda_i/z}) = qS_{0i} e^{\ln(q)\lambda_i/z}.$$

The Greek indices stand for the basis of flat vector fields corresponding to $1, H - \bar{\lambda}, \ldots, (H - \bar{\lambda})^m$.

Givental's oscillating integral solutions for S_{0i} are stationary phase expansions of the integrals

$$S_{0i} e^{\ln(q)\lambda_i/z} = (-2\pi z)^{-m/2} \int_{\Gamma_i \subset \{\sum T_j = \ln q\}} e^{F_i(T)/z} \omega$$

along m-cycles Γ_i through a specific critical point of $F_i(T)$ inside a m-dimensional $\mathbb{C}$-subspace of $\mathbb{C}^{m+1}$, where

$$F_i(T) = \sum_{j=0}^{m} (e^{T_j} + \lambda_j T_j).$$

The form ω is the restriction of $\mathrm{d}T_0 \wedge \cdots \wedge \mathrm{d}T_m$. To see that the integrals are actual solutions, notice that applying $D - \lambda_j$ to the integral has the same effect as multiplying the integrand by e^{T_j}.

There are $m+1$ possible critical points at which one can do a stationary phase expansion of S_{0i}. Let us write $P_i = Q_i + \bar{\lambda}$ for the solution to

$$\prod_{i=0}^{m} (X - \lambda_i) = q$$

with limit λ_i as $q \to 0$. For each i we need to choose the critical point $e^{T_j} = P_i - \lambda_j$ in order for the factor

$$\exp(u_i/z) := \exp\left(\left(\sum_{j=0}^{m} (P_i - \lambda_j + \lambda_j \ln(P_i - \lambda_j)) - \lambda_i \ln(q)\right)/z\right)$$

of S_{0i} to be well-defined in the limit $q \to 0$. Shifting the integral to the critical point and scaling coordinates by $\sqrt{-z}$ we find

$$S_{0i} = e^{u_i/z} \int \exp\left(-\sum_j (Q_i - \bar{\lambda}_j) \sum_{k=3}^{\infty} \frac{T_j^k (-z)^{(k-2)/2}}{k!}\right) \mathrm{d}\mu_i$$

for the conditional Gaussian distribution

$$\mathrm{d}\mu_i = (2\pi)^{-m/2} \exp\left(-\sum_j (Q_i - \bar{\lambda}_j) \frac{T_j^2}{2}\right) \omega.$$

The covariance matrices are given by

$$\sigma_i(T_k, T_l) = \frac{1}{\Delta_i} \begin{cases} -\prod_{j \notin \{k,l\}} (Q_i - \bar{\lambda}_j), & \text{for } k \neq l, \\ \sum_{m \neq k} \prod_{j \notin \{k,m\}} (Q_i - \bar{\lambda}_j), & \text{for } k = l. \end{cases}$$

From here we can see that the integral is symmetric in the $\bar{\lambda}_j$ and therefore we can write its asymptotics as $z \to 0$ completely in terms of data from A_{m+1}. Since odd moments of Gaussian distributions vanish we find that $e^{-u_i/z} S_{0i}$ is a power series in z with values in $\Delta_i^{-1/2} k_{A_{m+1}}[Q_i, \Delta_i^{-1}, \lambda]$.

So the entries of the R-matrix in the basis of normalized idempotents are given by

$$\Delta_k^{-1/2} \prod_{j \neq k} (D + \lambda_i - P_j) \; (e^{-u_i/z} S_{0i}).$$

Since $\frac{\mathrm{d}P_i}{\mathrm{d}q} = \frac{1}{\Delta_i}$ these entries are in

$$k_{A_{m+1}}[Q_0, \ldots, Q_m, \Delta_0^{-1/2}, \ldots, \Delta_m^{-1/2}, \lambda].$$

So, with the arguments from Section 3.1, the endomorphism R can be defined over $k_{\mathbb{P}^m}[\mathrm{disc}^{-1}]$.

We need to check that the R-matrix given in terms of oscillating integrals behaves correctly in the limit $q \to 0$. By definition, in this limit $P_i \to \lambda_i$. By symmetry it is enough to consider the 0-th column. Set $x_i = e^{T_i}$. Then

$$\begin{aligned}
\lim_{q \to 0} R_{j0} &\asymp \lim_{q \to 0} e^{-u_0/z} \Delta_0^{-1/2} \prod_{k \neq j} (zq \frac{\mathrm{d}}{\mathrm{d}q} + \lambda_j - \lambda_k) \; S_{00} \\
&= \lim_{q \to 0} \frac{e^{-u_0/z}}{\sqrt{\Delta_0}(-2\pi z)^{m/2}} \int e^{(\sum_k (e^{T_k} - (\lambda_0 - \lambda_k) T_k))/z + \sum_{k \neq j} T_k} \omega \\
&= \lim_{q \to 0} \frac{e^{-u_0/z}}{\sqrt{\Delta_0}(-2\pi z)^{m/2}} \int e^{(\sum\limits_{k \neq 0} (x_k - (\lambda_0 - \lambda_k) T_k) + \frac{q}{\prod_{k \neq 0} x_k})/z} \prod_{k \neq j} x_j \bigwedge_{k=1}^{m} \mathrm{d}T_k.
\end{aligned}$$

In the last step we have moved to the chart

$$x_0 = \frac{q}{\prod_{j \neq 0} x_j}.$$

Since in this chart $\lim_{q \to 0} x_0 = 0$, we have that R_{j0} vanishes unless $j = 0$. On the other hand in the limit $q \to 0$ the integral for R_{00} splits into one-dimensional integrals

$$\lim_{q \to 0} R_{00} \asymp \lim_{q \to 0} \frac{e^{-u_0/z}}{\sqrt{\Delta_0}(-2\pi z)^{m/2}} \prod_{k \neq 0} \int_0^{\infty} e^{(x - (\lambda_0 - \lambda_k) \ln(x))/z} \mathrm{d}x.$$

Let us temporarily set $z_k = -z/(\lambda_0 - \lambda_k)$. The prefactors also split into pieces in the limit and we calculate the factor corresponding to k to be

$$\begin{aligned}
\frac{e^{(1 - \ln(\lambda_0 - \lambda_k))/z_k}}{\sqrt{-2\pi z(\lambda_0 - \lambda_k)}} \int_0^{\infty} e^{(x - (\lambda_0 - \lambda_k) \ln(x))/z} \mathrm{d}x &= \frac{e^{(1 - \ln(1/z_k))/z_k}}{\sqrt{2\pi/z_k}} \Gamma\left(1 + \frac{1}{z_k}\right) \\
= \frac{e^{(1 - \ln(1/z_k))/z_k}}{\sqrt{2\pi z_k}} \Gamma\left(\frac{1}{z_k}\right) &\asymp \exp\left(\sum_{l=1}^{\infty} \frac{B_{2l}}{2l(2l-1)} \left(\frac{z}{\lambda_k - \lambda_0}\right)^{2l-1}\right),
\end{aligned}$$

using Stirling's approximation of the gamma function in the last step. So the product of the factors gives the expected limit (4) of R_{00} for $q \to 0$. This calculation gives a proof for the results [7] of Ionel on the main generating function used in [12] and [8] without having to use Harer's stability results.

4 $\mathbb{P}^m$ relations imply $(m+2)$-spin relations

We prove Theorem 1 in this section. As already mentioned, for this it is enough to show that, after the change $z \mapsto z\lambda^{-1}$, the $\mathbb{P}^m$-R-matrix converges to the A_{m+1}-R-matrix in the limit $\lambda \to \infty$. For this we have to compare the differential equations satisfied by the R-matrices.

Inserting the vector field corresponding to the hyperplane into (3) and using the divisor equation as in Section 3.2 gives the equation

$$[R_{\mathbb{P}^m}, \xi] + zq\frac{\mathrm{d}R_{\mathbb{P}^m}}{\mathrm{d}q} + zq\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}q}R_{\mathbb{P}^m} = 0, \tag{6}$$

where ξ denotes the diagonal matrix of quantum multiplication by $H - \bar{\lambda}$.

Lemma 1. *$R_{\mathbb{P}^m}(z/\lambda)$ admits a limit R for $\lambda \to \infty$. The matrix R satisfies*

$$[R, \xi] + z\frac{\mathrm{d}R}{\mathrm{d}t^1} + z\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}t^1}R = 0$$

$$z\frac{\mathrm{d}R}{\mathrm{d}z} + \sum_{\mu=1}^{m}\frac{m+2-\mu}{m+2}t^\mu\frac{\mathrm{d}R}{\mathrm{d}t^\mu} = 0$$

Proof. The $\mathbb{P}^m$-R-matrix satisfies the homogeneity property

$$z\frac{\mathrm{d}R_{\mathbb{P}^m}}{\mathrm{d}z} + (m+1)q\frac{\mathrm{d}R_{\mathbb{P}^m}}{\mathrm{d}q} + \sum_{i=0}^{m}\lambda_i\frac{\mathrm{d}R_{\mathbb{P}^m}}{\mathrm{d}\lambda_i} = 0.$$

So $R'(z) := R_{\mathbb{P}^m}(z/\lambda)$ written with the A_{m+1}-variables satisfies

$$(m+2)z\frac{\mathrm{d}R'}{\mathrm{d}z} + (m+1)\lambda\frac{\mathrm{d}R'}{\mathrm{d}\lambda} + \sum_{\mu=1}^{m}(m+2-\mu)t^\mu\frac{\mathrm{d}R'}{\mathrm{d}t^\mu} = 0.$$

The main differential equation satisfied by R' is

$$[R', \xi] + z\left(1 + \frac{t^1}{\lambda}\right)\frac{\mathrm{d}R'}{\mathrm{d}t^1} + z\left(1 + \frac{t^1}{\lambda}\right)\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}t^1}R' = 0.$$

From the expression of $R_{\mathbb{P}^m}$ in terms of oscillating integrals we know that the entries of the z^i-part R'_i of R' live in

$$\lambda^{-i}k_{A_{m+1}}[Q_0, \ldots, Q_m, \Delta_0^{-1/2}, \ldots, \Delta_m^{-1/2}, \lambda].$$

To show that the limit exists we need to show that λ occurs in no positive power. We will show this by induction by i. It certainly holds for $R'_0 = 1$. Since ξ is diagonal with pairwise distinct entries Q_j, the z^i-part of the differential equation determines the off-diagonal coefficients of R'_i in terms of R'_{i-1}. Because $\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}t^1}$ does not depend on λ, the off-diagonal coefficients of R'_i will admit the limit $\lambda \to \infty$. Since $\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}t^1}$ in general vanishes on the diagonal the diagonal coefficient of the z^{i+1}-part of the differential equation determines the diagonal of $\frac{\mathrm{d}R'_i}{\mathrm{d}t^1}$ from the off-diagonal entries of R'_i. Apart from a possible term constant in t^1 we therefore know that also the diagonal entries of R'_i admit the limit.

Let us consider such a possible ambiguity a_i. Since all products of Δ_j have dependence in t^1, the "denominator" of a_i can only be a power of λ

less than i. However then a_i cannot possibly satisfy the homogeneity. By induction therefore the limit R exists. The properties of R easily follow from the corresponding ones of R'. □

By inserting the vector field $\frac{\partial}{\partial t^1}$ into (3) and similar arguments as in the proof of Lemma 1 one can show the following lemma.

Lemma 2. *The A_{m+1}-R-matrix is uniquely determined from the differential equation*

$$[R_{A_{m+1}}, \xi] + z\frac{\mathrm{d}R_{A_{m+1}}}{\mathrm{d}t^1} + z\Psi^{-1}\frac{\mathrm{d}\Psi}{\mathrm{d}t^1}R_{A_{m+1}} = 0,$$

the homogeneity

$$z\frac{\mathrm{d}R_{A_{m+1}}}{\mathrm{d}z} + \sum_{\mu=1}^{m}\frac{m+2-\mu}{m+2}t^\mu\frac{\mathrm{d}R_{A_{m+1}}}{\mathrm{d}t^\mu} = 0$$

and that the entries of the z-series coefficients of $R_{A_{m+1}}$ should lie in

$$k_{A_{m+1}}[Q_0, \dots, Q_m, \Delta_0^{-1/2}, \dots, \Delta_m^{-1/2}].$$

The lemmas imply that the modified $\mathbb{P}^m$-R-matrix contains only non-positive powers of λ and the part constant in λ equals the A_{m+1}-R-matrix. Therefore the A_{m+1}-relations are contained in the modified $\mathbb{P}^m$-relations as the λ^0-part, and we have completed the proof of Theorem 1.

5 Equivalence of relations

We want to give a proof of Theorem 2 in this section. So we will consider the CohFTs in the Airy limit, i.e. with all t^μ but $t := t^1$ set to zero. In this limit the metric becomes $\eta(X^i, X^j) = \delta_{i+j,m}$, the quantum product stays semisimple and the Euler vector field for the A_m-singularity

$$E = \frac{m+1}{m+2}t\frac{\partial}{\partial t}$$

is a multiple of X.

Rewriting (6) for the $\mathbb{P}^m$-R-matrix $\tilde{R}_{\mathbb{P}^m} = \Psi R_{\mathbb{P}^m}\Psi^{-1}$ written in flat coordinates gives

$$[\tilde{R}_{\mathbb{P}^m}, \xi] - zqL_E\tilde{R}_{\mathbb{P}^m} + zq\tilde{R}_{\mathbb{P}^m}\mu = 0,$$

where ξ is multiplication by E in flat coordinates and $\mu = -(L_E\Psi)\Psi^{-1}$.

We need to find a series φ in t and an R-matrix R sending the modified A_{m+1}-theory to equivariant $\mathbb{P}^m$:

$$\tilde{R}_{\mathbb{P}^m}(z) = R(z)\tilde{R}_{A_{m+1}}(z\varphi).$$

We know that $\tilde{R}_{A_{m+1}}$ satisfies

$$[\tilde{R}_{A_{m+1}}, \xi] + zL_E\tilde{R}_{A_{m+1}} - z\tilde{R}_{A_{m+1}}\mu = 0$$

and the weighted homogeneity condition

$$\left(z\frac{\mathrm{d}}{\mathrm{d}z}+L_E\right)\tilde{R}_{A_{m+1}}+[\mu,\tilde{R}_{A_{m+1}}]=0.$$

Putting these together we find that R must satisfy

$$\begin{aligned}0=[R,\xi]-zqL_ER+zq\frac{L_E\varphi}{\varphi}R\mu\\&+\frac{1}{\varphi}\left(q+\varphi-q\frac{L_E\varphi}{\varphi}\right)R[\tilde{R}_{A_{m+1}}(z\varphi),\xi]\tilde{R}^{-1}_{A_{m+1}}(z\varphi).\end{aligned}$$

Lemma 3. *The series $\tilde{R}_{A_{m+1}}\xi\tilde{R}^{-1}_{A_{m+1}}$ is not a polynomial in z.*

Because of the lemma and the homogeneity of $\tilde{R}_{A_{m+1}}$ we see that in order for R to exist in the limit disc $\to 0$ the function φ has to satisfy

$$q+\varphi-q\frac{L_E\varphi}{\varphi}=0$$

or equivalently

$$-q^{-1}=\varphi^{-1}+L_E\varphi^{-1}.$$

There is a unique solution φ^{-1} to this differential equation. Concretely, we have

$$\varphi^{-1}=\lambda^{-1}\sum_{i=0}^{\infty}\frac{m+2}{m+2+i(m+1)}\left(-\frac{t}{\lambda}\right)^i.$$

Since it is not necessary for the proof of Theorem 2, we will prove Lemma 3 in Section 6.

Let us from now on assume that φ is this solution. Then the differential equation for R spells

$$[R,\xi]-zqL_ER+zq\frac{L_E\varphi}{\varphi}R\mu=0. \tag{7}$$

The following lemma implies that the matrix $\tilde{R}_{\mathbb{P}^m}(z)\tilde{R}^{-1}_{A_{m+1}}(z\varphi)$ does not have any poles in t and this concludes the proof of Theorem 2.

Lemma 4. *For any solution $R(z)$ of (7) of the form*

$$R(z)=\sum_{i=0}^{\infty}(R^i_{jk})z^i=1+O(z),$$

for Laurent series R^i_{jk} in t, actually all the R^i_{jk} have to be polynomials.

Proof. The matrices ξ and μ can be explicitly calculated

$$\xi_{jk}=t\frac{m+1}{m+2}\delta_{j,k+1}(-t)^{\delta_{0,j}},\qquad \mu_{jk}=\frac{2j-m}{2(m+2)}\delta_{j,k},$$

where all indices are understood modulo $(m+1)$.

Assume that we have already constructed R^{i-1} and its entries have no negative powers in t. Looking at the z^i-part of (7) gives expressions for $R^i_{j(k+1)}\xi_{(k+1)k}-\xi_{j(j-1)}R^i_{(j-1)k}$ as power series with no poles in t. From here we see that if we can determine the R^i_{j0} as power series with no poles, then the other entries are given by

$$R^i_{jk} \equiv (-t)^{\delta_{k>j}} R^i_{(j-k)0}$$

modulo terms with no poles in t, determined from R^{i-1}. The exponent $\delta_{k>j}$ is 1 for $k > j$ and 0 otherwise.

From the z^{i+1}-part of (7) we then get expressions with no poles in t for

$$(m+1)t\frac{\mathrm{d}R^i_{j0}}{\mathrm{d}t} + jR^i_{j0},$$

thus determining all R^i_{j0} but R^i_{00} up to a constant. Therefore all the R^i_{jk} are polynomials in t. □

Remark 5. The derivation in this section would have worked the same if q was any other invertible power series in t.

6 Higher dimensions

We would like to show that for $m > 1$ there is no pair of function φ and matrix power series $R(z)$, both well-defined in the limit disc $\to 0$, such that

$$\tilde{R}_{\mathbb{P}^m}(z) = R(z)\tilde{R}_{A_{m+1}}(z\varphi), \tag{8}$$

where again $\tilde{R}_* = \Psi R_* \Psi^{-1}$. We will need to assume that that φ is well-defined in the Airy limit. Then we can use the discussion from Section 5 to derive that φ is of the form

$$\varphi = \lambda + c_0\lambda^0 + c_{-1}\lambda^{-1} + \cdots ,$$

where the c_i are independent of λ and c_{-1} in the Airy limit becomes a constant multiple of $(t^1)^2$. For the uniqueness of φ we needed Lemma 3.

Proof of Lemma 3. Recall that we have to show that $P := \tilde{R}_{A_{m+1}}\xi\tilde{R}^{-1}_{A_{m+1}}$ is not a polynomial in z. From the differential equation for $\tilde{R}_{A_{m+1}}$ we obtain a differential equation for P.

$$[P,\xi] = z^2\frac{\mathrm{d}P}{\mathrm{d}z} - z[P,\mu]$$

By definition we also have the initial condition $P|_{z=0} = \xi$. Write $P = \xi + zP_1 + z^2P_2 + \cdots$. The homogeneity condition for $\tilde{R}_{A_{m+1}}$ implies that the only nonzero entries of P_i are at the $(i-1)$-th diagonal, where by this we mean the entries on j-th row, k-th column such that $k - j \equiv i - 1 \pmod{m+1}$.

Assume we have shown that $P_i \neq 0$ has a nonzero entry on the $(i-1)$-th diagonal row. Recalling the proof of Lemma 4 we see that essentially the differences of two subsequent entries in the i-th diagonal of P_{i+1} are a multiple of an entry on the $(i-1)$-th diagonal of P_i. Since the absolute value of any entry of μ is less than $\frac{1}{2}$, all of these multiples are nonzero. Therefore it is impossible for all entries on the i-th diagonal of P_{i+1} to be zero. The lemma follows by induction. □

To show that there is no suitable intermediate R-matrix R it will be enough to consider the z^1-term of (8). It says

$$\tilde{r}_{\mathbb{P}^m} = r + \varphi\tilde{r}_{A_{m+1}},$$

where r_* stands for the z^1-term of R_*. Since $\tilde{r}_{\mathbb{P}^m}$ has no negative powers in λ, the λ^{-1}-terms on the right hand side have to cancel. However the bottom-left coefficient of $\tilde{r}_{A_{m+1}}$ has a pole in the discriminant. Since for $m > 2$ the coefficient c_{-1} cannot be a multiple of the discriminant for degree reasons, in this case r has to have a pole in the discriminant. Contradiction.

It remains to look at the case $m = 2$. Here it is similarly enough to show that there is one coefficient in the R-matrix with a second order pole in the discriminant in order to derive a contradiction. We look at the coefficient r_{20} calculated from the oscillating integral of Section 3.1. We need to calculate the z^1-coefficient of the asymptotic expansion of

$$\sum_Q \frac{1}{\sqrt{2\pi}\Delta}\int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2} - x^3\sqrt{-z}\frac{Q}{\Delta^{3/2}} - \frac{x^4}{4}(-z)\frac{1}{\Delta^2}\right) \mathrm{d}x,$$

where we sum over roots Q of the polynomial defining the singularity and here $\Delta = 3Q^2 + 2t^2$. Expanding the Gaussian integral we find the coefficient to be equal to

$$-\sum_Q \frac{15}{2}\frac{Q^2}{\Delta^4} + \sum_Q \frac{3}{\Delta^3}.$$

It is straightforward to check that the first summand equals

$$-\frac{15}{2}\frac{-2(2t^2)^3 + 27(t^1)^2}{(-4(2t^2)^3 - 27(t^1)^2)^2},$$

whereas the second term has only a first order pole in the discriminant.

A Givental's localization calculation

We want to recall Givental's localization calculation [4], which proves that the CohFT from equivariant $\mathbb{P}^m$ can be obtained from the trivial theory via a specific R-matrix action. We first recall localization in the space of stable maps to $\mathbb{P}^m$ in Section A.1. Next, in Section A.2 we group the localization contributions according to the dual graph of the source curve. We collect identities following from the string and dilaton equation in Section A.3 before applying them to finish the computation in Section A.4.

A.1 Localization in the space of stable maps

Let $T = (\mathbb{C}^*)^{m+1}$ act diagonally on $\mathbb{P}^m$. The equivariant Chow ring of a point and $\mathbb{P}^m$ are given by

$$\begin{aligned} A_T^*(\mathrm{pt}) &\cong \mathbb{Q}[\lambda_0, \dots, \lambda_m] \\ A_T^*(\mathbb{P}^m) &\cong \mathbb{Q}[H, \lambda_0, \dots, \lambda_m] / \prod_{i=0}^{m} (H - \lambda_i), \end{aligned}$$

where H is a lift of the hyperplane class. Furthermore, let η be the equivariant Poincaré pairing.

There are $m+1$ fixed points $p_0, \dots, p_m$ for the T-action on $\mathbb{P}^m$. The characters of the action of T on the tangent space $T_{p_i}\mathbb{P}^m$ are given by $\lambda_i - \lambda_j$ for $j \neq i$. Hence the corresponding equivariant Euler class e_i is given by

$$e_i = \prod_{j \neq i} (\lambda_i - \lambda_j).$$

The equivariant class e_i also serves as the inverse of the norms of the equivariant (classical) idempotents

$$\phi_i = e_i^{-1} \prod_{j \neq i} (H - \lambda_j).$$

The virtual localization formula [6] implies that the virtual fundamental class can be split into a sum

$$[\overline{M}_{g,n}(\mathbb{P}^m, d)]_T^{vir} = \sum_X \iota_{X,*} \frac{[X]_T^{vir}}{e_T(N_{X,T}^{vir})}$$

of contributions of fixed loci X. Here $N_{X,T}^{vir}$ denotes the virtual normal bundle of X in $\overline{M}_{g,n}(\mathbb{P}^m, d)$ and e_T the equivariant Euler class. Because of the denominator, the fixed point contributions are only defined after localizing by the elements $\lambda_0, \dots, \lambda_m$. By studying the $\mathbb{C}^*$-action on deformations and obstructions of stable maps, $e_T(N_{X,T}^{vir})$ can be computed explicitly.

The fixed loci can be labeled by certain decorated graphs. These consist of

- a graph (V, E),
- an assignment $\zeta : V \to \{p_0, \dots, p_m\}$ of fixed points,
- a genus assignment $g : V \to \mathbb{Z}_{\geq 0}$,
- a degree assignment $d : E \to \mathbb{Z}_{>0}$,
- an assignment $p : \{1, \dots n\} \to V$ of marked points,

such that the graph is connected and contains no self-edges, two adjacent vertices are not assigned to the same fixed point and we have

$$g = h^1(\Gamma) + \sum_{v \in V} g(v), \qquad d = \sum_{e \in E} d(e).$$

A vertex $v \in V$ is called stable if $2g(v) - 2 + n(v) > 0$, where $n(v)$ is the number of outgoing edges at v.

The fixed locus corresponding to a graph is characterized by the condition that stable vertices $v \in V$ of the graph correspond to contracted genus $g(v)$ components of the domain curve, and that edges $e \in E$ correspond to multiple covers of degree $d(e)$ of the torus fixed line between two fixed points. Such a fixed locus is isomorphic to a product of moduli spaces of curves

$$\prod_{v \in V} \overline{M}_{g(v),n(v)}$$

up to a finite map.

For a fixed locus X corresponding to a given graph the Euler class $e_T(N^{vir}_{X,T})$ is a product of factors corresponding to the geometry of the graph

$$e_T(N^{vir}_{X,T}) = \prod_{v,\text{ stable}} \frac{e(\mathbb{E}^* \otimes T_{\mathbb{P}^m,\zeta(v)})}{e_{\zeta(v)}} \prod_{\text{nodes}} \frac{e_\zeta}{-\psi_1 - \psi_2} \prod_{\substack{g(v)=0 \\ n(v)=1}} (-\psi_v) \prod_e \mathrm{Contr}_e. \quad (9)$$

In the first product $\mathbb{E}^*$ denotes the dual of the Hodge bundle, $T_{\mathbb{P}^m,\zeta(v)}$ is the tangent space of $\mathbb{P}^m$ at $\zeta(v)$, and all bundles and Euler classes should be considered equivariantly. The second product is over nodes forced onto the domain curve by the graph. They correspond to stable vertices together with an outgoing edge, or vertices v of genus 0 with $n(v) = 2$. With ψ_1 and ψ_2 we denote the (equivariant) cotangent line classes at the two sides of the node. For example, the equivariant cotangent line class ψ at a fixed point p_i on a line mapped with degree d to a fixed line is more explicitly given by

$$-\psi = \frac{\lambda_j - \lambda_i}{d},$$

where p_j is the other fixed point on the fixed line. The explicit expressions for the terms in the second line of (9) can be found in [6], but will play no role for us. It is only important that they only depend on local data.

A.2 General procedure

We set W to be $A^*_T(\mathbb{P}^m)$ with all equivariant parameters localized. For $v_1, \dots, v_n \in W$ the (full) CohFT $\Omega_{g,n}$ from equivariant $\mathbb{P}^m$ is defined by

$$\begin{aligned} &\Omega^p_{g,n}(v_1, \dots, v_n) \\ &= \sum_{d,k=0}^{\infty} \frac{q^d}{k!} \varepsilon_* \pi_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(v_i) \prod_{i=n+1}^{n+k} \mathrm{ev}_i^*(p) \cap [\overline{M}_{g,n+k}(\mathbb{P}^m, d)]^{vir} \right), \end{aligned} \quad (10)$$

where p is a point on the formal Frobenius manifold, ε forgets the last k markings and π forgets the map. We want to calculate the push-forward via virtual localization. In the end we will arrive at the formula of the R-matrix action as described in Section 2.3. In the following we will systematically suppress the dependence on p in the notation.

We start by remarking that for each localization graph for (10) there exists a dual graph of $\overline{M}_{g,n}$ corresponding to the topological type of the stabilization of a generic source curve of that locus. What gets contracted under the stabilization maps are trees of rational curves. There are three types of these unstable trees:

1. those which contain one of the n markings and are connected to a stable component,
2. those which are connected to two stable components and contain none of the n markings and
3. those which are connected to one stable component but contain none of the n markings.

These give rise to series of localization contributions and we want to record those, using the fact that they already occur in genus 0.

Let W' be an abstract free module over the same base ring as W with a basis $w_0, \dots, w_m$ labeled by the fixed points of the T-action on $\mathbb{P}^m$. The type 1 contributions are recorded by

$$\tilde{R}^{-1} = \sum_i \tilde{R}_i^{-1} w_i \in \operatorname{Hom}(W, W')[\![z]\!],$$

the homomorphism valued power series such that

$$\tilde{R}_i^{-1}(v) = \eta(e_i \phi_i, v) + \sum_{d,k=0}^{\infty} \frac{q^d}{k!} \sum_{\Gamma \in G^1_{d,k,i}} \frac{1}{\operatorname{Aut}(\Gamma)} \operatorname{Contr}_\Gamma(v)$$

where $G^1_{d,k,i}$ is the set of localization graphs for $\overline{M}_{0,2+k}(\mathbb{P}^m, d)$ such that the first marking is at a valence 2 vertex at fixed point i and $\operatorname{Contr}_\Gamma(v)$ is the contribution for graph Γ for the integral

$$\int_{\overline{M}_{0,2+k}(\mathbb{P}^m,d)} \frac{e_i}{-z-\psi_1} \operatorname{ev}_2^*(v) \prod_{l=3}^{2+k} \operatorname{ev}_l^*(p).$$

We define the integral in the case $(d,k) = (0,0)$ to be zero and will do likewise for other integrals over non-existing moduli spaces.

The type 2 contributions are recorded by the bivector

$$\tilde{V} = \sum_i \tilde{V}^{ij} w_i \otimes w_j \in W'^{\otimes 2}[\![z, w]\!]$$

which is defined by

$$V^{ij} = \sum_{d,k=0}^{\infty} \frac{q^d}{k!} \sum_{\Gamma \in G^2_{d,k,i,j}} \frac{1}{\operatorname{Aut}(\Gamma)} \operatorname{Contr}_\Gamma,$$

where $G^2_{d,k,i,j}$ is the set of localization graphs for $\overline{M}_{0,2+k}(\mathbb{P}^m, d)$ such that the first and second marking are at valence 2 vertices at fixed points i and j, respectively, and $\operatorname{Contr}_\Gamma$ is the contribution for graph Γ for the integral

$$\int_{\overline{M}_{0,2+k}(\mathbb{P}^m,d)} \frac{e_i e_j}{(-z-\psi_1)(-w-\psi_2)} \prod_{l=3}^{2+k} \operatorname{ev}_l^*(p).$$

Finally, the type 3 contribution is a vector

$$\tilde{T} = \sum_i \tilde{T}_i w_i \in W'[\![z]\!]$$

which is defined by

$$\tilde{T}_i = p + \sum_{d,k=0}^{\infty} \frac{q^d}{k!} \sum_{\Gamma \in G^3_{d,k,i}} \frac{1}{\mathrm{Aut}(\Gamma)} \mathrm{Contr}_\Gamma$$

where $G^3_{d,k,i}$ is the set of localization graphs for $\overline{M}_{0,1+k}(\mathbb{P}^m, d)$ such that the first marking is at a valence 2 vertex at fixed point i and $\mathrm{Contr}_\Gamma(v)$ is the contribution for graph Γ for the integral

$$\int_{\overline{M}_{0,1+k}(\mathbb{P}^m,d)} \frac{e_i}{-z-\psi} \prod_{l=2}^{1+k} \mathrm{ev}_l^*(p).$$

With these contributions we can write the CohFT already in a form quite similar to the reconstruction formula. Let $\omega_{g,n}$ be the n-form on W' which vanishes if w_i and w_j for $i \neq j$ are inputs, which satisfies

$$\omega_{g,n}(w_i, \dots, w_i) = \frac{e(\mathbb{E}^* \otimes T_{\mathbb{P}^m, p_i})}{e_i} = e_i^{g-1} \prod_{j \neq i} c_{\lambda_j - \lambda_i}(\mathbb{E})$$

and which is for $n = 0$ defined similarly as in Example 1. We have

$$\Omega^p_{g,n}(v_1, \dots, v_n) = \sum_\Gamma \frac{1}{\mathrm{Aut}(\Gamma)} \xi_* \left(\prod_v \sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon_* \omega_{g_v, n_v + k}(\dots) \right), \qquad (11)$$

where we put

1. $\tilde{R}^{-1}(\psi)(v_i)$ into the argument corresponding to marking i,
2. a half of $\tilde{V}(\psi_1, \psi_2)$ into an argument corresponding to a node and
3. $\tilde{T}(\psi)$ into all additional arguments.

We will still need to apply the string and dilaton equation in order to make $\tilde{T}(\psi)$ to be a multiple of ψ^2, like the corresponding series in the reconstruction, and then relate the series to the R-matrix.

A.3 String and Dilaton Equation

We want to use the string and dilaton equation to bring a series

$$\sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon_* \left(\prod_{i=1}^{n} \frac{1}{-x_i - \psi_i} \prod_{i=n+1}^{n+k} Q(\psi_i) \right), \qquad (12)$$

where $\varepsilon : \overline{M}_{g,n+k} \to \overline{M}_{g,n}$ is the forgetful map and $Q = Q_0 + zQ_1 + z^2 Q_2 + \cdots$ is a formal series, into a canonical form.

By the string equation, (12) is annihilated by

$$\mathcal{L}' = \mathcal{L} + \sum_{i=1}^{n} \frac{1}{x_i},$$

where $\mathcal{L}$ is the string operator

$$\mathcal{L} = \frac{\partial}{\partial Q_0} - Q_1 \frac{\partial}{\partial Q_0} - Q_2 \frac{\partial}{\partial Q_1} - Q_3 \frac{\partial}{\partial Q_2} - \cdots .$$

Moving along the string flow for some time $-u$, i.e. applying $e^{t\mathcal{L}'}|_{t=-u}$, to (12) gives

$$\sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon_* \left(\prod_{i=1}^{n} \frac{e^{-\frac{u}{x_i}}}{-x_i - \psi_i} \prod_{i=n+1}^{n+k} Q'(\psi_i) \right),$$

for a new formal series $Q' = Q'_0 + zQ'_1 + z^2 Q'_2 + \cdots$. In the case that

$$u = \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \prod_{i=3}^{2+k} Q(\psi_i),$$

which we will assume from now on, the new series Q' will satisfy $Q'_0 = 0$ since by the string equation $\mathcal{L}u = 1$ and therefore applying $e^{t\mathcal{L}}|_{t=-u}$ to u gives on the one hand zero and on the other hand the definition of u with Q replaced by Q', and for dimension reasons this is a nonzero multiple of Q'_0.

Next, by applying the dilaton equation we can remove the linear part from the series Q'_0

$$\sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon_* \left(\prod_{i=1}^{n} \frac{1}{-x_i - \psi_i} \prod_{i=n+1}^{n+k} Q(\psi_i) \right) = \sum_{k=0}^{\infty} \frac{\Delta^{\frac{2g-2+n+k}{2}}}{k!} \varepsilon_* \left(\prod_{i=1}^{n} \frac{e^{-\frac{u}{x_i}}}{-x_i - \psi_i} \prod_{i=n+1}^{n+k} Q''(\psi_i) \right), \quad (13)$$

where $Q'' = Q' - Q'_1 z$ and

$$\Delta^{\frac{1}{2}} = (1 - Q'_1)^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\overline{M}_{0,3+k}} \prod_{i=4}^{3+k} Q(\psi_i).$$

We will also need identities in the degenerate cases $(g,n) = (0,2)$ and $(g,n) = (0,1)$. In the first case, there is the identity

$$\frac{1}{-z-w} + \sum_{k=1}^{\infty} \frac{1}{k!} \int_{\overline{M}_{0,2+k}} \frac{1}{-z-\psi_1} \frac{1}{-w-\psi_2} \prod_{i=3}^{2+k} Q(\psi_i) = \frac{e^{-u/z+-u/w}}{-z-w}. \quad (14)$$

In order to see that (14) is true, we use that the left hand side is annihilated by $\mathcal{L} + \frac{1}{z} + \frac{1}{w}$ in order to move from Q to Q' via the string flow and notice that there all of the integrals vanish for dimension reasons. Similarly, there is the identity

$$1 - \frac{Q(z)}{z} - \frac{1}{z} \sum_{k=2}^{\infty} \frac{1}{k!} \int_{\overline{M}_{0,1+k}} \prod_{i=2}^{1+k} Q(\psi_i) = e^{-u/z} \left(1 - \frac{Q'(z)}{z} \right) = e^{-u/z} \left(\Delta^{-\frac{1}{2}} - \frac{Q''(z)}{z} \right), \quad (15)$$

which can be proven like the previous identity by using that the left hand side is annihilated by $\mathcal{L} + \frac{1}{z}$.

We define the functions u_i and $(\Delta_i/e_i)^{\frac{1}{2}}$ for $i \in \{0, \dots, m\}$ to be the u and $\Delta^{\frac{1}{2}}$ at the points $Q = \tilde{T}_i$ from the previous section.

A.4 Expressing localization series in terms of Frobenius structures

We apply (13) to (11) and obtain

$$\Omega^p_{g,n}(v_1, \dots, v_n) = \sum_\Gamma \frac{1}{\mathrm{Aut}(\Gamma)} \xi_* \left(\prod_v \sum_{k=0}^\infty \frac{1}{k!} \varepsilon_* \omega'_{g_v, n_v+k}(\dots) \right), \qquad (16)$$

where we put

1. $R^{-1}(\psi)(v_i)$ into the argument corresponding to marking i,
2. a half of $V(\psi_1, \psi_2)$ into an argument corresponding to a node and
3. $T(\psi)$ into all additional arguments.

Here R^{-1}, V and T are defined exactly as $\tilde{R}^{-1}$, $\tilde{V}$ and $\tilde{T}$ but with the replacement

$$\frac{e_i}{-x-\psi} \rightsquigarrow \frac{e_i e^{-\frac{u_i}{x}}}{-x-\psi}$$

made at the factors we put at the ends of the trees. The form $\omega'_{g,n}$ satisfies

$$\omega'_{g,n}(w_i, \dots, w_i) = \Delta_i^{\frac{2g-2+n}{2}} e_i^{-\frac{n}{2}} \prod_{j \neq i} c_{\lambda_j - \lambda_i}(\mathbb{E}).$$

We now want to compute R^{-1}, V and T in terms of the homomorphism valued series $S^{-1}(z) \in \mathrm{Hom}(W, W')[\![z]\!]$ with w_i-component

$$\begin{aligned} S_i^{-1}(z) &= \langle \frac{e_i \phi_i}{-z-\psi}, - \rangle \\ &:= \eta(e_i \phi_i, -) + \sum_{d,k=0}^\infty \frac{q^d}{k!} \int_{\overline{M}_{0,2+k}(\mathbb{P}^m, d)} \frac{\mathrm{ev}_1^*(e_i \phi_i)}{-z-\psi_1} \mathrm{ev}_2^*(-) \prod_{j=3}^{2+k} \mathrm{ev}_j^*(p). \end{aligned}$$

We start by computing S^{-1} via localization. Using that in genus zero the Hodge bundle is trivial we find that at the vertex with the first marking we need to compute integrals exactly as in (14), where the first summand stands for the case that the vertex is unstable and the k-summand stands for the case that the vertex is stable with k trees of type 3 and one tree of type 1. Applying (14) we obtain

$$S_i^{-1}(z) = e^{-\frac{u_i}{z}} R_i^{-1}(z).$$

Using the short-hand notation

$$\left\langle \frac{v_1}{x_1-\psi}, \frac{v_2}{x_2-\psi}, \frac{v_3}{x_3-\psi} \right\rangle$$
$$:= \sum_{d,k=0}^{\infty} \frac{q^d}{k!} \int_{\overline{M}_{0,3+k}(\mathbb{P}^m,d)} \frac{\mathrm{ev}_1^* v_1}{x_1-\psi_1} \frac{\mathrm{ev}_2^* v_2}{x_2-\psi_2} \frac{\mathrm{ev}_3^* v_3}{x_3-\psi_3} \prod_{i=4}^{3+k} \mathrm{ev}_i^*(p)$$

for genus zero Gromov-Witten invariants and applying the string equation, we can also write

$$S_i^{-1}(z) = -\frac{1}{z}\langle \frac{e_i\phi_i}{-z-\psi}, \mathbf{1}, -\rangle.$$

We have by the identity axiom and WDVV equation

$$\langle \frac{e_i\phi_i}{-z-\psi}, \frac{e_j\phi_j}{-w-\psi}, \mathbf{1}\rangle = \langle \frac{e_i\phi_i}{-z-\psi}, \frac{e_j\phi_j}{-w-\psi}, \bullet\rangle\langle \bullet, \mathbf{1}, \mathbf{1}\rangle$$
$$= \langle \frac{e_i\phi_i}{-z-\psi}, \mathbf{1}, \bullet\rangle\langle \bullet, \mathbf{1}, \frac{e_j\phi_j}{-w-\psi}\rangle,$$

where in the latter two expressions the $\bullet$ should be filled with η^{-1}, so that

$$\frac{S_i^{-1}(z)\eta^{-1}S_j^{-1}(w)^t}{-z-w}$$
$$= \frac{\eta(e_i\phi_i, e_j\phi_j)}{-z-w} + \sum_{d,k=0}^{\infty} \frac{q^d}{k!} \int_{\overline{M}_{0,2+k}(\mathbb{P}^m,d)} \frac{\mathrm{ev}_1^*(e_i\phi_i)}{-z-\psi_1} \frac{\mathrm{ev}_2^*(e_j\phi_j)}{-w-\psi_2} \prod_{l=3}^{2+k} \mathrm{ev}_l^*(p). \tag{17}$$

We compute the right hand side via localization. There are two cases in the localization depending on whether the first and second marking are at the same or a different vertex. In the first case we apply (14) at this common vertex and obtain the total contribution

$$\frac{e_i\delta_{ij}e^{-\frac{u_i}{z}-\frac{u_j}{w}}}{-z-w},$$

which includes the unstable summand. In the other case, we apply (14) at the two vertices and obtain

$$e^{-\frac{u_i}{z}-\frac{u_j}{w}}V^{ij}(z,w).$$

So all together

$$V^{ij}(z,w) = \frac{R_i^{-1}(z)\eta^{-1}R_j^{-1}(w)^t - e_i\delta_{ij}}{-z-w}.$$

Finally we express T in terms of R by computing

$$S_i^{-1}(z)\mathbf{1} = e_i - \frac{1}{z}\sum_{d,k=0}^{\infty} \frac{q^d}{k!} \int_{\overline{M}_{0,1+k}(\mathbb{P}^m,d)} \frac{\mathrm{ev}_1^*(e_i\phi_i)}{-z-\psi_1} \prod_{j=2}^{1+k} \mathrm{ev}_j^*(p)$$

via localization. Applying (15) at the first marking we find that

$$S_i^{-1}(z)\mathbf{1} = e^{-\frac{u_i}{z}} \left(\Delta_i^{-\frac{1}{2}} e_i^{\frac{1}{2}} - \frac{T_i(z)}{z} \right).$$

So

$$T(z) = z \left(\sum_i \Delta_i^{-\frac{1}{2}} e_i^{\frac{1}{2}} w_i - R^{-1}(z)\mathbf{1} \right).$$

By (16) the underlying TQFT of $\Omega^p_{g,0}$ is given by

$$\sum_i \Delta_i^{g-1}.$$

This implies that the Δ_i need to be the inverses of the norms of the idempotents for the quantum product of equivariant $\mathbb{P}^m$ (because these are pairwise different). Since $\tilde{T}$ vanishes at $(p, q) = 0$, Δ_i is e_i^{-1} at $(p, q) \to 0$. Therefore we can identify W' with W by mapping w_i to $\sqrt{\Delta_i/e_i}$ times the idempotent element which coincides with ϕ_i at $(p, q) = 0$. The previous results then say exactly that Ω^p is obtained from the CohFT ω' by the action of the R-matrix R. In turn, Example 4 implies that ω' is obtained from the TQFT by the action of an R-matrix which is diagonal in the basis of idempotents and has entries

$$\exp\left(\sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} \sum_{j \neq i} \left(\frac{z}{\lambda_j - \lambda_i} \right)^{2i-1} \right).$$

We still need to check that R satisfies the quantum differential equation and has the correct limit (4) as $q \to 0$. By considering (17) as $w + z \to 0$ we see that $S^{-1}(z)$ satisfies the symplectic condition, i.e. its inverse $S(z)$ is the adjoint with respect to η of $S^{-1}(-z)$. More explicitly the evaluation of $S(z)$ at the ith normalized idempotent is the vector

$$\left\langle \frac{\sqrt{e_i}\phi_i}{z - \psi}, \eta^{-1} \right\rangle.$$

By the genus 0 topological recursion relations for any flat vector field X

$$z \left\langle X, \frac{\sqrt{e_i}\phi_i}{z - \psi}, \eta^{-1} \right\rangle = \langle X, \eta^{-1}, \bullet \rangle \left\langle \bullet, \frac{\sqrt{e_i}\phi_i}{z - \psi} \right\rangle,$$

where again η^{-1} should be inserted at the $\bullet$. Therefore S satisfies the quantum differential equation

$$zXS(z) = X \star S(z),$$

where on the left hand side the action of vector fields and the right hand side quantum multiplication is used.

At $q = 0$, we can check that R becomes the identity matrix and therefore the R-matrix of Ω^p becomes the R-matrix of the CohFT ω', which has the correct limit (4).

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Paper C

Relations in the Tautological Ring and Frobenius Manifolds near the Discriminant

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Relations in the Tautological Ring and Frobenius Manifolds near the Discriminant

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Abstract

For generically semisimple cohomological field theories pole cancellation in the Givental-Teleman classification implies relations between classes in the tautological ring of the moduli space of curves. For the theory of the A_2-singularity these are known to be equivalent to Pixton's generalized Faber-Zagier relations. We show that the relations from any other semisimple cohomological field theory can be written in terms of Pixton's relations. This gives large evidence for the conjecture that Pixton's relations are all relations between tautological classes.

As part of the proof, we study the structure of an N-dimensional generically semisimple Frobenius manifold near smooth points of the non-semisimple locus, giving a local description modeled on the Frobenius manifold corresponding to the $A_2 \times A_1^{N-2}$-singularity, and give criteria for extending generically semi-simple Frobenius manifolds to cohomological field theories.

1 Introduction

The tautological rings $RH^*(\overline{M}_{g,n})$ are certain subrings of the cohomology rings $H^*(\overline{M}_{g,n})$ of the Deligne-Mumford moduli space $\overline{M}_{g,n}$ of stable curves of arithmetic genus g with n markings. Starting from the 80s with Mumford's seminal article [13], they have been studied extensively. However, their structure is still not completely understood: While there is an explicit set of generators parametrized by decorated graphs, the set of relations between the generators is not known. On the other hand, Pixton's set [16] of generalized Faber-Zagier relations gives a well-tested conjectural description for this set of relations. Another conjectural description had been given by Faber's Gorenstein conjecture but it is now known to be false in general [15].

In [14] the relations of Pixton have been shown to arise in the computation of Witten's 3-spin class via the Givental-Teleman classification of semisimple cohomological field theories (CohFTs). The formula that Pandharipande-Pixton-Zvonkine obtain for Witten's 3-spin class has the form of a limit $\phi \to 0$ of a Laurent series in a variable ϕ whose coefficients are tautological classes. The existence of the limit implies cancellation between tautological classes such that no poles in ϕ are left in the end. These relations between tautological classes, after adding relations directly following from them, give exactly the relations of Pixton.

As noted in [14], the limit $\phi \to 0$ can be viewed as approaching a non-semisimple point on the Frobenius manifold corresponding to Witten's 3-spin class. In particular, the same procedure can be applied to get relations from other generically semisimple CohFTs but it is not clear how the relations from different CohFTs relate to each other. In [10] first comparison results have been proven: The relations from the equivariant Gromov-Witten theory of $\mathbb{P}^1$ are equivalent to the relations from the 3-spin theory and in general the relations from equivariant $\mathbb{P}^{N-1}$ imply the $(N+1)$-spin relations.

The main result of this article, Theorem 3.3.6, is that, for any CohFT to which this procedure applies, we obtain the same set of relations. Thus the relations of Pixton are the universal relations necessary in order for the Givental-Teleman classification to admit non-semisimple limits. Theorem 3.3.6 can also be used to relate more geometric relations to Pixton's relations (see e.g. [2]).

Before attacking Theorem 3.3.6, we prove a structure result, Theorem 2.3.10, about Frobenius manifolds near a smooth point of the discriminant locus of non-semisimple points. Essentially we show that there is a nice set of local coordinates and local vector fields, which is modeled on the simplest example of the 3-spin theory (extended to the correct dimension using trivial theories).

Using Theorem 2.3.10 we give a criterion (Theorem 3.4.1) when a generically semisimple Frobenius manifold can locally be extended to a CohFT. Its proof first locally identifies points and tangent vectors of the given Frobenius manifold and of the 3-spin Frobenius manifold. Under this identification we show that an extension is obtained from the 3-spin CohFT by the action of an R-matrix and a shift, both of which are holomorphic along the discriminant. Theorem 3.3.6 essentially follows by noticing that for formal reasons these (invertible) operations preserve the corresponding tautological relations.

In this paper we work over $\mathbb{C}$ and with the tautological ring in cohomology. It is actually more natural to define the tautological ring in Chow and everything in this paper works equally well in Chow if the Givental-Teleman reconstruction is proven in Chow for the relevant CohFTs.

Plan of the paper

In Section 2 we first recall basic properties of Frobenius manifolds, define the discriminant and then prove Theorem 2.3.10 about the local structure of semisimple Frobenius manifolds near a smooth point on the discriminant. In Section 3 we start by recalling the definition of cohomological field theories and the statement of the Givental-Teleman classification. After that, in Section 3.3, we discuss the tautological ring and the relations resulting from the classification. In Section 3.4 we prove Theorem 3.4.1 about the extension of locally semi-simple Frobenius manifolds. We discuss in Section 3.5 how its proof implies Theorem 3.3.6 on the comparison of tautological relations. In Section 3.6, we shortly consider the problem of finding a global extension theorem similar to Theorem 3.4.1. Afterwards, in Section 3.7 we study two examples, which illustrate obstructions to directly generalizing our results. In the final Section 3.8 we show that certain other relations obtained from the equivariant Gromov-Witten theory of toric targets can also be expressed in terms of Pixton's relations.

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The author has learned about the method of obtaining relations by studying a CohFT near the discriminant from D. Zvonkine at the conference *Cohomology of the moduli space of curves* organized by the *Forschungsinstitut für Mathematik at ETH Zürich* in 2013. D. Zvonkine there also expressed an idea why we should obtain the same relations from different CohFTs. Together with S. Shadrin he studied them in comparison to relations obtained from degree considerations. See [10] for a short discussion about how these are related.

The results on the extension of a Frobenius manifold to a CohFT is motivated from discussions with M. Kazarian, T. Milanov and D. Zvonkine at the workshop *Geometric Invariants and Spectral Curves* at the *Lorentz Center* in Leiden.

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2 Frobenius manifolds

2.1 Definition and basic properties

Frobenius manifolds have been introduced by Dubrovin [3]. They naturally arise when studying genus zero Gromov-Witten theory. Let us begin by recalling their basic properties in the following slightly redundant definition.

Definition 2.1.1. An N-dimensional (complex, even) Frobenius manifold is a 4-tuple $(M, \eta, A, \mathbf{1})$, consisting of

- M, a complex, connected manifold of dimension N,
- a nonsingular metric $\eta \in \Gamma(\mathrm{Sym}^2(T^*M))$,
- a tensor $A \in \Gamma(\mathrm{Sym}^3(T^*M))$,
- a vector field $\mathbf{1} \in \Gamma(TM)$,

satisfying the following properties:

- A commutative, associative product $\star$ on TM, with unit $\mathbf{1}$, is defined by setting for local vector fields X and Y that

  $$\eta(X \star Y, Z) = A(X, Y, Z)$$

  for any local vector field Z.

- The metric η is flat and $\mathbf{1}$ is an η-flat vector field.
- Locally around each point there exist *flat* coordinates t_α such that the metric and the unit vector field are constant when written in the basis of the corresponding local vector fields $\frac{\partial}{\partial t_\alpha}$.
- Locally on M there exists a holomorphic function Φ called *potential* such that
$$A\left(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta}, \frac{\partial}{\partial t_\gamma}\right) = \frac{\partial^3 \Phi}{\partial t_\alpha \partial t_\beta \partial t_\gamma}.$$

2.2 Discriminant and semisimplicity

Let U be a chart of an N-dimensional Frobenius manifold with a basis $e_1, \dots, e_N$ of flat vector fields. There is a trace map Tr taking vector fields on U to holomorphic functions on U defined by setting for any $p \in M$ and vector field X that $\mathrm{Tr}(X)(p)$ is the trace of the linear map on T_pU given by $\star$-multiplication by $X|_p$. We define a *discriminant* function disc of U by
$$\mathrm{disc} = \det(\mathrm{Tr}(e_i e_j)) \in \mathcal{O}_U.$$
The function disc is not independent of the choice of flat vector fields: If A changes from one flat basis to another, the discriminant changes by the constant $\det(A)^2$. However, this means that at least the discriminant locus $\{\mathrm{disc} = 0\}$ is well-defined. Over any point p of U there exists a nilpotent element in T_pU if and only if p lies in the discriminant locus.

We say that a Frobenius manifold M is (generically) *semisimple* if the discriminant is not identically zero. We call a point in M semisimple if it does not lie in the discriminant locus.

If M is semisimple, near any semisimple point we can choose a basis $\frac{\partial}{\partial u_i}$ of orthogonal idempotents and we use the notation Δ_i^{-1} for their norms. Then $\Delta_i^{\frac{1}{2}} \frac{\partial}{\partial u_i}$ define *normalized idempotents*. As the notation suggests, the vector fields $\frac{\partial}{\partial u_i}$ commute and we can integrate them locally near semisimple points to give the *canonical coordinates* u_i.

Lemma 2.2.1. *For an N-dimensional semisimple Frobenius manifold M, locally around any (possibly not semisimple) point, the orthogonal idempotents extend to meromorphic sections of π^*TM, where $\pi : \tilde{M} \to M$ is a finite holomorphic map with ramification at most along the discriminant. Furthermore, the meromorphic sections can have poles at most along the discriminant.*

Proof. Take a local basis e_μ of flat vector fields and consider their minimal polynomials f_μ, which have holomorphic coefficients. On a finite ramified cover $\tilde{M}$ we can single out holomorphic roots $\zeta_{\mu,i}$ for each minimal polynomial. If none of the differences $\zeta_{\mu,i} - \zeta_{\mu,j}$ vanishes identically, we can define idempotent meromorphic vector fields on $\tilde{M}$ by the Lagrange interpolation polynomials
$$\prod_{j \neq i} \frac{e_\mu - \zeta_{\mu.j}}{\zeta_{\mu,i} - \zeta_{\mu,j}}. \tag{1}$$

On the other hand, if the difference $\zeta_{\mu,i} - \zeta_{\mu,j}$ vanishes identically, the numerator $P = \prod_{j \neq i}(e_\mu - \zeta_{\mu.j})$ of the Lagrange interpolation polynomial still makes sense and is on the one hand non-zero because otherwise adding up the Galois conjugates of P would give an equation for e_μ of degree lower than the minimal polynomial, but on the other hand its square vanishes. So because of semisimplicity of M this case is actually impossible.

Some idempotent vector fields of (1) might coincide or not be orthogonal to each other, but we know because the e_μ gave a basis at each tangent space that the vector fields from (1) span the tangent space of a generic point. By removing duplicate idempotents and by suitably taking differences we can extract a basis of orthogonal idempotents for the tangent space of a generic point.

Given a semisimple point p, we can choose an element of the tangent space whose minimal polynomial has order N, i.e. a linear combination of orthogonal idempotents with pairwise distinct coefficients. Therefore also the minimal polynomial of the corresponding flat vector field has order N. Its discriminant does not vanish at p and therefore its roots are not ramified along p. By construction, the corresponding idempotents recover the idempotents at T_pM and are in particular holomorphic in p. □

Example 2.2.2. The Givental-Saito theory of the A_2 singularity, which appears in the study of Witten's 3-spin class (it is mirror symmetric), is about a two-dimensional Frobenius manifold and motivates Theorem 2.3.10. As a manifold it is isomorphic to $\mathbb{C}^2$ with coordinates t_0, t_1 and its points correspond to versal transformations

$$\frac{x^3}{3} - t_1 x + t_0$$

of the A_2-singularity $\frac{x^3}{3}$. In the basis $\frac{\partial}{\partial t_0}$, $\frac{\partial}{\partial t_1}$ the metric η is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the potential is

$$\Phi(t_0, t_1) = \frac{1}{2} t_0^2 t_1 + \frac{1}{24} t_1^4.$$

Therefore $\frac{\partial}{\partial t_0}$ is the unit and the only interesting quantum product is

$$\frac{\partial}{\partial t_1} \star \frac{\partial}{\partial t_1} = t_1 \frac{\partial}{\partial t_0}.$$

Hence on a two-fold cover of $\mathbb{C}^2$ ramified along the discriminant locus $\{t_1 = 0\}$ we can define the meromorphic idempotents

$$\epsilon_\pm = \pm \frac{1}{2\sqrt{t_1}} \frac{\partial}{\partial t_1} + \frac{1}{2} \frac{\partial}{\partial t_0}.$$

A choice of corresponding canonical coordinates is given by

$$u_\pm = t_0 \pm \frac{2}{3} t_1^{\frac{3}{2}}.$$

Notice that we can get back to the flat vector fields by

$$\frac{\partial}{\partial t_0} = \epsilon_+ + \epsilon_-$$

and

$$\frac{\partial}{\partial t_1} = \left(\frac{3}{4}(u_+ - u_-)\right)^{\frac{1}{3}} (\epsilon_+ - \epsilon_-).$$

2.3 Local structure near the discriminant

We now want to analyze in more detail the structure of a semisimple Frobenius manifold near the discriminant locus. The results are summarized in Theorem 2.3.10.

For this we start with a neighborhood M of a smooth point p of the discriminant locus D of an N-dimensional Frobenius manifold. We might need to shrink this neighborhood a finite number of times but by abuse of notation we will keep the name M.

Because of smoothness of $p \in D$, on a smaller M there exists a positive integer k and flat coordinates $t_1, \dots, t_N$ such that there is a kth root $t_D = \sqrt[k]{\mathrm{disc}}$ that can be expanded near p as

$$t_D = t_1 + O((t_1, t_2, \dots, t_N)^2).$$

We will use $t_D, t_2, \dots, t_N$ as alternative local coordinates and study the order in t_D, i.e. order of vanishing along D, of various data of M. By shrinking M further we can assume that D is the vanishing locus of t_D.

By Lemma 2.2.1, in order to define idempotents we will also need to allow for vector fields whose coefficients are convergent Puisseux series in t_D. So in the following the order in t_D can also be fractional.

Lemma 2.3.1. *No nonzero idempotent can have positive order in t_D.*

Proof. Let X be an idempotent with positive order m. Then $X = X^2$ would have order at least $2m$. However for positive m we have $m < 2m$. □

Lemma 2.3.2. *There is a choice of the flat coordinates $t_2, \dots, t_N$ such that for every i, when we write*

$$\frac{\partial}{\partial u_i} = c_1 \frac{\partial}{\partial t_D} + \sum_{\mu=2}^{N} c_\mu \frac{\partial}{\partial t_\mu},$$

the function c_1 has the minimal t_D-order $-m_i$ out of $\{c_1, \dots, c_N\}$.

Proof. Since under the coordinate change $t'_\mu = t_\mu + \alpha_\mu t_D$ for $\alpha_\mu \in \mathbb{C}$, $\mu \in \{2, \dots, N\}$ we have

$$\frac{\partial}{\partial t_1} = \frac{\partial}{\partial t'_1} + \sum_{\mu=2}^{N} \alpha_\mu \frac{\partial}{\partial t'_\mu}, \qquad \frac{\partial}{\partial t_\mu} = \frac{\partial}{\partial t'_\mu},$$

there is a dense set of suitable coordinate transforms for any i. □

We will from now on assume that we have made such a choice of flat coordinates.

Lemma 2.3.3. *At least one of the norms Δ_i^{-1} of the orthogonal idempotents has negative order in t_D.*

Proof. Assume on the contrary that the norms of all idempotents extend to D. Pick an idempotent $\frac{\partial}{\partial u_i}$ with most negative order $-m_i$. Then the element

$$X = t_D^{m_i} \frac{\partial}{\partial u_i}$$

extends to D but by assumption $\eta(X, X)$ vanishes there. Since the metric is nonsingular we can find a vector field Y defined in a neighborhood of a point on D, such that $\eta(X, Y)$ has order zero along D. Writing Y as

$$Y = \sum_j c_j \frac{\partial}{\partial u_j},$$

we see that $t_D^{m_i} c_i$ has also order zero.

Therefore there is an $m > 0$ such that $Y' := t_D^m Y$ can be written as

$$Y' = \sum_j c'_j \frac{\partial}{\partial u_j},$$

where all c'_j have non-negative order and at least one c'_j has order zero in t_D. On the other hand, we know that Y' vanishes on D.

The set of all Y' satisfying these conditions is closed under taking powers and taking linear combinations which do not make all order zero c'_j vanish. Using these operations we can make, up to reordering, the idempotents, $c'_1, \ldots, c'_k$ for $k \leq N$ to be equal to 1 up to higher order terms in t_D, and also make all other c'_j vanish in D. Furthermore, we can assume that $c'_1 = 1$.

Now for large l either

$$Y'^l - \sum_{j=1}^{k} \frac{\partial}{\partial u_j}$$

vanishes along D or we can replace Y' by $Y'^{2l} - Y'^l$ times a suitable (negative) power of t_D, thereby making k smaller. For $k = 1$ always the first case occurs.

So using induction we find that there is an idempotent element which vanishes along D. This is in contradiction to Lemma 2.3.1. □

Lemma 2.3.4. *The order $-m$ of c_1 as in Lemma 2.3.2 agrees for all idempotents with negative order. The order $-m'$ of the norms Δ_i^{-1} of these idempotents also agrees and is negative.*

Proof. For idempotents $\frac{\partial}{\partial u_i}$ and $\frac{\partial}{\partial u_j}$ with negative order, by Lemma 2.3.2 the part of lowest order in t_D of the $\frac{\partial}{\partial t_D}$-component of the commutator

$$\left[\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}\right] = 0$$

can be calculated from the lowest order parts of the $\frac{\partial}{\partial t_D}$-components of $\frac{\partial}{\partial u_i}$ and $\frac{\partial}{\partial u_j}$, and these will not agree if they have different orders in t_D.

For the second part we compare the order of both sides of the identity

$$\frac{\partial \Delta_j^{-1}}{\partial u_i} = \frac{\partial \Delta_i^{-1}}{\partial u_j}$$

from [3, Lecture 3]. If Δ_j has positive order m', the left hand side has order $-m'-m-1$. Therefore in this case Δ_i also needs to have order m'. By Lemma 2.3.3 we are done. $\square$

Lemma 2.3.5. *The number of idempotents with negative order is equal to 2.*

Proof. To exclude the possibility that the number of idempotents with negative order is at least 3 we consider the Darboux-Egoroff equations [3, Lecture 3][1]

$$\frac{\partial \gamma_{ij}}{\partial u_k} = \gamma_{ik}\gamma_{kj} \tag{2}$$

for i, j, k corresponding to a triple of such idempotents. Here the γ_{ij} are the rotation coefficients

$$\gamma_{ij} = \sqrt{\Delta_j}\frac{\partial}{\partial u_j}\Delta_i^{-1/2}.$$

The order on both sides of (2) is $-2m-2$. Let c_i, c_j, c_k and d_i, d_j, d_k be the lowest order coefficients in t_D of the $\frac{\partial}{\partial t_D}$-component of the idempotents and their norms, respectively. So the lowest order terms of γ_{ij} and (2) are

$$-\frac{m'}{2}d_j^{-1/2}c_j d_i^{1/2}$$

and

$$-\frac{m'}{2}(-m-1)c_k d_j^{-1/2}c_j d_i^{1/2} = \frac{m'^2}{4}d_k^{-1/2}c_k d_i^{1/2}d_j^{-1/2}c_j d_k^{1/2},$$

respectively. So we need to have

$$\frac{m'^2}{4} - \frac{m'm}{2} - \frac{m'}{2} = 0,$$

which does not hold since by Lemma 2.3.3 $m' > 0$ and we must have $m' \leq m$.

Because in general the sum of all idempotents is the identity, we conclude that there are exactly two idempotents with negative order. $\square$

We will from now on assume that $\frac{\partial}{\partial u_1}$ and $\frac{\partial}{\partial u_2}$ are these two idempotents.

Lemma 2.3.6. *The vector fields*

$$t_D^m\left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right), \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_{\geq 3}} \tag{3}$$

give a basis of the tangent space at every point of M.

[1] In [3] always the assumption of the existence of an Euler vector field is made. We here only use Equation 3.70a, which holds also without this assumption.

Proof. All of the vector fields extend to D and the first of them does not converge to the second because that would imply that the second vector field is both idempotent and nilpotent, i.e. zero. They therefore give a basis of the tangent space at generic points in D. Since they also give a basis for any tangent space outside D, they give a basis for the tangent space at every point of M. $\square$

Lemma 2.3.7. *The order m' of Δ_1 and Δ_2 is equal to m.*

Proof. The norm of the first vector field of (3) vanishes on D. This together with the fact that the metric is nonsingular implies that

$$t_D^m(\Delta_1^{-1} - \Delta_2^{-1})$$

must have order zero in t_D. Therefore Δ_1 and Δ_2 must have order m. $\square$

Lemma 2.3.8. *After possibly shrinking M there exists a flat vector field X such that the vector fields $1, X, \ldots, X^{N-1}$ span the tangent space at every point of M.*

Proof. We write any X in the basis (3). We have found a suitable X when in a neighborhood of p the first coefficient is not zero and the other coefficients are pairwise different and nonzero. However if these conditions were false for any flat X, flat vector fields would only generate a proper linear subspace at the tangent space of p. $\square$

Let X be as in Lemma 2.3.8. There is an equation

$$\prod_{i=1}^{N}(X - \zeta_i) = 0$$

and at least one of the differences $\zeta_i - \zeta_j$ has positive order in t_D. The idempotents are then given by

$$\frac{\partial}{\partial u_i} = \prod_{j \neq i} \frac{X - \zeta_j}{\zeta_i - \zeta_j} \tag{4}$$

Because $1, X, \ldots, X^{N-1}$ span the tangent space at a generic point of D, the products $\prod_{j\neq i}(X - \zeta_j)$ all have order 0 and because of (4) all roots of the characteristic polynomial are distinct along D apart from ζ_1 and ζ_2.

We also know that when restricted to D all roots of the characteristic polynomial are distinct apart from ζ_1 and ζ_2. Since the coefficients of the characteristic polynomial are holomorphic, we find that $\zeta_{\geq 3}$ are holomorphic, as well as $\zeta_1 + \zeta_2$ and $\zeta_1\zeta_2$. Therefore $2m$ is a positive integer.

The vector fields $\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}$, $\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$ and $\frac{\partial}{\partial u_{\geq 3}}$ all commute and all of them but $\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$ are holomorphic. It follows that by correctly choosing integration constants we can have that $u_1 - u_2$ has order $(m+1)$ in t_D. Furthermore by Lemma 2.3.6, $t_D^{-m-1}(u_1 - u_2)$ does not vanish on any point of D. We conclude that we can choose a root $(u_1 - u_2)^{1/(m+1)}$ that has order one in t_D and is holomorphic in p. We thus can work with $(u_1 - u_2)^{1/(m+1)}$ instead of t_D.

Lemma 2.3.9. *If for all flat vector fields X a genus one potential $\mathrm{d}G(X)$ of the Frobenius manifold extends to the discriminant, the order $-m$ has to be equal to $-\frac{1}{2}$.*

Proof. The general formula (see [5]) for a genus one potential G is given by

$$\mathrm{d}G = \frac{1}{48}\sum_i \mathrm{d}\log(\Delta_i) + \frac{1}{2}\sum_i r_{ii}\mathrm{d}u_i, \tag{5}$$

where the functions ${r_{ii}}^2$, determined up to an integration constant, satisfy

$$\mathrm{d}r_{ii} = \frac{1}{4}\sum_j \frac{\partial\log(\Delta_j)}{\partial u_i}\frac{\partial\log(\Delta_i)}{\partial u_j}(\mathrm{d}u_j - \mathrm{d}u_i).$$

Let us consider the lowest order term in t_D of

$$\frac{\partial G}{\partial u_1} - \frac{\partial G}{\partial u_2} = \frac{1}{24}\sum_i \frac{\partial\log(\Delta_i)}{\partial(u_1-u_2)} + \frac{1}{2}(r_{11}-r_{22}).$$

For this we will only need to care about the $i=1$ and $i=2$ terms of the sum, whose lowest order terms are both equal to

$$\frac{m}{m+1}\frac{1}{u_1-u_2}.$$

For the $\mathrm{d}(u_1-u_2)$-component of $\mathrm{d}r_{ii}$ for $i\in\{1,2\}$, we get the lowest order term from the $j=3-i$ summand, which is

$$(-1)^{i+1}\frac{m^2}{4(m+1)^2}\frac{1}{(u_1-u_2)^2}.$$

In total the lowest order terms of $\frac{\partial G}{\partial u_1} - \frac{\partial G}{\partial u_2}$ are

$$\frac{2m}{24(m+1)}\frac{1}{u_1-u_2} - \frac{m^2}{4(m+1)^2}\frac{1}{u_1-u_2}.$$

This term only vanishes when $m=\frac{1}{2}$.

Therefore if $m\neq\frac{1}{2}$, at p the genus one potential $\mathrm{d}G$ with $\frac{\partial}{\partial t_D}$ inserted is not holomorphic along D. □

In total, we have shown the following result.

Theorem 2.3.10. *Let M be a semisimple Frobenius manifold. In a neighborhood U of a smooth point of the discriminant of M, there exists a positive half-integer m such that*

- *all but two idempotents $\frac{\partial}{\partial u_1}$, $\frac{\partial}{\partial u_2}$ extend holomorphically to the discriminant in U,*
- *for a suitable choice of integration constants, there is a holomorphic root $(u_1-u_2)^{1/(m+1)}$, and its vanishing locus describes the discriminant in U,*

[2] They correspond to the diagonal entries of the linear part of the R-matrix.

- *The vector fields*

$$(u_1 - u_2)^{m/(m+1)} \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2} \right), \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}, \frac{\partial}{\partial u_{\geq 3}}$$

  extend holomorphically to the discriminant and span the tangent space at every point of U. The first of these vector fields spans the space of nilpotent tangent vectors at p.

Furthermore, if in addition for any flat vector field X, for a genus one potential G, the function $\mathrm{d}G(X)$ extends to the discriminant, the half-integer m has to be equal to $\frac{1}{2}$.

3 Cohomological Field Theories

3.1 Definitions

Let $\overline{M}_{g,n}$ be the moduli space of stable, connected, at most nodal algebraic curves of arithmetic genus g with n markings. It is a smooth DM-stack of dimension $3g-3+n$. Let $M_{g,n}$ be the open substack of smooth pointed curves. Forgetting a marking and gluing along markings induce the tautological maps

$$\begin{aligned} \overline{M}_{g,n+1} &\to \overline{M}_{g,n}, \\ \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} &\to \overline{M}_{g_1+g_2,n_1+n_2}, \\ \overline{M}_{g,n+2} &\to \overline{M}_{g+1,n}. \end{aligned}$$

Cohomological field theories were first introduced by Kontsevich and Manin in [11] to formalize the structure of classes from Gromov-Witten theory. Let V be an N-dimensional $\mathbb{C}$-vector space and η a nonsingular bilinear form on V.

Definition 3.1.1. A cohomological field theory (CohFT) Ω on (V,η) is a system

$$\Omega_{g,n} \in H^*(\overline{M}_{g,n}) \otimes (V^*)^{\otimes n}$$

of multilinear forms with values in the cohomology ring of $\overline{M}_{g,n}$ satisfying the following properties:

Symmetry $\Omega_{g,n}$ is symmetric in its n arguments

Gluing The pull-back of $\Omega_{g,n}$ via the gluing map

$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}$$

is given by the direct product of Ω_{g_1,n_2+1} and Ω_{g_2,n_2+1} with the bivector η^{-1} inserted at the two points glued together. Similarly for the gluing map $\overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$ the pull-back of $\Omega_{g,n}$ is given by $\Omega_{g-1,n+2}$ with η^{-1} inserted at the two points glued together.

Unit There is a special element $\mathbf{1} \in V$ called the *unit* such that

$$\Omega_{g,n+1}(v_1, \ldots, v_n, \mathbf{1})$$

is the pull-back of $\Omega_{g,n}(v_1, \ldots, v_n)$ under the forgetful map and

$$\Omega_{0,3}(v, w, \mathbf{1}) = \eta(v, w).$$

Definition 3.1.2. A CohFT whose classes are only multiples of the fundamental class is called a *topological field theory* (TQFT).

The definition of CohFTs can be also generalized to families of CohFTs over a ground ring. We will use the following non-standard definition.

Definition 3.1.3. Let $e_1, \dots, e_N$ be a basis of V. A *convergent CohFT* Ω on V is a CohFT defined over the ring of holomorphic functions of an open neighborhood U of $0 \in V$ such that for all $g \geq 0$, all $\alpha_1, \dots, \alpha_n \in V$ and all $\mathbf{t} = t_1 e_1 + \cdots + t_N e_N$ we have

$$\Omega_{g,n}|_{\mathbf{t}}(\alpha_1, \dots, \alpha_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Omega_{g,n+k}|_0(\alpha_1, \dots, \alpha_n, \mathbf{t}, \dots, \mathbf{t}). \qquad (6)$$

We can define from any usual CohFT a convergent CohFT by using (6), under the assumption that the sum converges in a neighborhood of 0.

Definition 3.1.4. The *underlying Frobenius manifold* of a convergent CohFT Ω is as a manifold the neighborhood U of $0 \in V$. At every point of U the tangent space is identified with V by sending the vector field $\frac{\partial}{\partial t_\mu}$ at every point to e_μ. With this identification η defines the metric, $\mathbf{1}$ defines the unit vector field and $\Omega_{0,3}$ defines the symmetric tensor A.

Remark 3.1.5. Restricting to the origin, we see that every CohFT determines a Frobenius algebra. This operation restricts to a bijection between TQFTs and Frobenius algebras of dimension N.

Using the underlying Frobenius manifold, for any convergent CohFT we can define the quantum product on V (depending on a point in U), the discriminant function, semisimplicity, semisimple points and the discriminant locus.

Example 3.1.6. Given an N-dimensional (convergent) CohFT Ω and some $c \in \mathbb{C}^*$, we can define an $(N+1)$-dimensional (convergent) CohFT Ω': If V is the underlying vector space of Ω, then $V \oplus \langle v \rangle$ will be the underlying vector space for Ω'. The nonsingular bilinear form η' on $V \oplus \mathbb{C}$ is defined via $\eta'(v,v) = c$, $\eta'(\alpha, v) = 0$ and $\eta'(\alpha,\beta) = \eta(\alpha,\beta)$, where $\alpha, \beta \in V$ and η is the nonsingular bilinear form of V. The CohFT Ω' is then defined by multilinearity from setting

$$\Omega'_{g,n}(\alpha_1, \dots, \alpha_n) = \Omega_{g,n}(\alpha_1, \dots, \alpha_n),$$

if all α_i lie in V, imposing the condition that Ω' vanishes if one argument is a multiple of v and another argument lies in V, and setting

$$\Omega'_{g,n}(v, \dots, v) = c^{1-g}.$$

Finding the right definition in the remaining case $n = 0$ and checking the axioms of a CohFT is left as an exercise to the reader.

Notice that v will be an idempotent element for the quantum product and that this operation therefore preserves semisimplicity.

3.2 Reconstruction

The (upper half of the) *symplectic loop group* corresponding to a vector space V with nonsingular bilinear form η is the group of endomorphism valued power series $V[\![z]\!]$ such that the *symplectic condition* $R(z)R^t(-z) = 1$ holds. Here R^t is the adjoint of R with respect to η. There is an action of this group on the space of all CohFTs based on a fixed semisimple Frobenius algebra structure of V. The action is named after Givental because he has introduced it on the level of arbitrary genus Gromov-Witten potentials.

Given a CohFT $\Omega_{g,n}$ and such an endomorphism R, the new CohFT $R\Omega_{g,n}$ takes the form of a sum over dual graphs Γ

$$R\Omega_{g,n}(v_1, \dots, v_n) = \sum_{\Gamma} \frac{1}{\mathrm{Aut}(\Gamma)} \xi_* \left(\prod_v \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Omega_{g_v, n_v+k}(\dots) \right), \tag{7}$$

where $\xi : \prod_v \overline{M}_{g_v,n_v} \to \overline{M}_{g,n}$ is the gluing map of curves of topological type Γ from their irreducible components, $\pi : \overline{M}_{g_v,n_v+k} \to \overline{M}_{g_v,n_v}$ forgets the last k markings and we still need to specify what is put into the arguments of $\prod_v \Omega_{g_v,n_v+k_v}$. Instead of only allowing vectors in V to be put into $\Omega_{g,n}$ we will allow for elements of $V[\![\psi_1, \dots, \psi_n]\!]$ where ψ_i acts on the cohomology of the moduli space by multiplication with the ith cotangent line class.

- Into each argument corresponding to a marking of the curve, put $R^{-1}(\psi)$ applied to the corresponding vector.
- Into each pair of arguments corresponding to an edge put the bivector
$$\frac{R^{-1}(\psi_1)\eta^{-1}R^{-1}(\psi_2)^t - \eta^{-1}}{-\psi_1 - \psi_2} \in V^{\otimes 2}[\![\psi_1, \psi_2]\!],$$
where one has to substitute the ψ-classes at each side of the normalization of the node for ψ_1 and ψ_2. By the symplectic condition this is well-defined.
- At each of the additional arguments for each vertex put
$$T(\psi) := \psi(\mathrm{Id} - R^{-1}(\psi))\mathbf{1},$$
where ψ is the cotangent line class corresponding to that vertex. Since $T(z) = O(z^2)$ the above k-sum is finite.

The following reconstruction result (on the level of potentials) has been first proposed by Givental [6].

Theorem 3.2.1 ([17]). *The R-matrix action is free and transitive on the space of semisimple CohFTs based on a given Frobenius algebra.*

Furthermore, given a convergent semisimple CohFT Ω, locally around a semisimple point, the element R of the symplectic loop group, taking the TQFT corresponding to the Frobenius algebra to Ω, satisfies the following differential equation of one-forms when written in a basis of normalized idempotents

$$[R(z), d\mathbf{u}] + z\Psi^{-1}d(\Psi R(z)) = 0. \tag{8}$$

Here $\mathbf{u}$ *is the diagonal matrix filled with the canonical coordinates* u_i *corresponding to the idempotents and* Ψ *is the basis change from the basis of normalized idempotents to a flat basis.*

Remark 3.2.2. The differential equation (8) makes sense for any Frobenius manifold. In general it defines R only up to right multiplication by a diagonal matrix whose entries are of the form $\exp(a_1 z + a_3 z^3 + \cdots)$, where the a_i are constants on the Frobenius manifold [7]. If the further condition of homogeneity with respect to an Euler vector field is imposed on R, there is a unique solution.

Remark 3.2.3. Teleman's proof relies heavily on topological results (Mumford's conjecture/Madsen-Weiss theorem) and it is therefore not known if the same classification result also holds in general when we work in Chow instead of cohomology. It is still known that the statement is also in some cases such as for the equivariant Gromov-Witten theory of a toric variety.

Remark 3.2.4. Formula (5) for a genus one potential

$$\mathrm{d}G(X) = \int_{\overline{M}_{1,1}} \Omega_{1,1}(X)$$

is a special case of the reconstruction.

Let us make the local structure of the reconstruction formula a bit more concrete for later use. We can decompose any endomorphism F of V into a collection of linear forms

$$F = \sum_i F^i \tilde{\epsilon}_i,$$

where $\tilde{\epsilon}_i$ is the ith normalized idempotent element and we will use the formula

$$\omega_{g,n}(\tilde{\epsilon}_{a_1}, \ldots, \tilde{\epsilon}_{a_n}) = \begin{cases} \sum_i \Delta_i^{g-1}, & \text{if } n = 0, \\ \Delta_{a_1}^{\frac{2g-2+n}{2}}, & \text{if } a_1 = \cdots = a_n, \\ 0, & \text{else,} \end{cases}$$

where the Δ_i are the inverses of the norms of the idempotents. Then we can rewrite (7) to

$$R\Omega_{g,n}(v_1, \ldots, v_n) = \sum_{\Gamma, c} \frac{1}{\mathrm{Aut}(\Gamma, c)} \xi_* \left(\prod_v C_{v,c(v)}(\ldots) \right), \tag{9}$$

where c is a coloring of the vertices of Γ by a color in the set $\{1, \ldots, N\}$ and the local contribution $C_{v,i}$ at a vertex v of genus g, with n markings and of color i is an n-form taking power series in z as inputs and is given by

$$C_{v,i}(\alpha_1, \ldots, \alpha_n) = \sum_{k=0}^{\infty} \frac{\Delta_i^{\frac{2g-2+n+k}{2}}}{k!} \pi_* \left(\prod_{j=1}^{m} \alpha_j(\psi_j) \prod_{j=n+1}^{n+k} \psi_j (\mathrm{Id}^i - (R^{-1}(\psi_j))^i) \mathbf{1}_\Omega \right).$$

The still missing arguments in (9), which correspond to preimages of the marked points and nodes in the normalization, are to be filled with the coordinates corresponding to the coloring of the vectors and bivectors also used in (7).

3.3 Relations in the tautological ring

The tautological subrings $R^*(\overline{M}_{g,n})$ can be compactly defined [4] as the smallest system of subrings $R^*(\overline{M}_{g,n}) \subseteq H^*(\overline{M}_{g,n})$ stable under push-forward under the tautological maps as described in Section 3.1. Each tautological ring is finitely generated [8] and a ring of generators has been formalized into the strata algebra $\mathcal{S}_{g,n}$ [16]. As the name suggests, the strata algebra is generated by strata of $\overline{M}_{g,n}$ (corresponding to dual graphs) decorated with Morita-Mumford-Miller κ-classes and ψ-classes. Pushforwards and pullbacks along the gluing and forgetful morphisms can be lifted to homomorphisms of the corresponding strata algebras satisfying the push-pull formula, ... Relations in the tautological ring are elements of the kernel of the natural projection $\mathcal{S}_{g,n} \to R^*(\overline{M}_{g,n})$.

Consider a semisimple, N-dimensional convergent CohFT Ω defined in a neighborhood U of $0 \in V$. Let $D \subset U$ be the discriminant locus. By the reconstruction formula described in Section 3.2 for each point outside D in U we can find an R-matrix such that Ω is given by applying the action of R to the underlying TQFT.

We obtain relations in the tautological ring by studying the behavior along D. On the one hand the reconstruction gives functions that might have singularities along the discriminant locus.[3] On the other hand we know that we get back the original CohFT by projecting from the strata algebra to the tautological ring. Therefore we obtain vector spaces of relations with values in $\mathcal{O}_{U\setminus D}/\mathcal{O}_U$.[4] By choosing a basis of $\mathcal{O}_{U\setminus D}/\mathcal{O}_U$ we obtain a vector space of relations.

Definition 3.3.1. The vector space of tautological relations associated to the convergent CohFT Ω is defined as the smallest system of ideals of $\mathcal{S}_{g,n}$ which is stable under push-forwards via the gluing and forgetful morphisms and contains the relations from cancellations of singularities in the reconstruction of Ω, that we have just defined.

Example 3.3.2. For the 2-dimensional, (convergent) CohFT corresponding to Witten's 3-spin class, in [14] it is proven that the ideal of relations coincides with the relations of Pixton [16], which are conjectured [16] to be all relations between tautological classes.

Example 3.3.3. In [10] it is shown that the ideal of relations of the Gromov-Witten theory of equivariant projective space $\mathbb{P}^{N-1}$ contains the relations for Witten's $(N+1)$-spin class.

Example 3.3.4. In [10] it is also shown that the set of relations for equivariant $\mathbb{P}^1$ and Witten's 3-spin class coincide.

Remark 3.3.5. For nonequivariant $\mathbb{P}^1$ the theory does not apply since the Frobenius manifold is semisimple at all points. There is a different way of how to extract relations in this case, which we will study in Section 3.8.

Our main result is the following.

Theorem 3.3.6. *For any two semisimple convergent CohFTs which are not semisimple at all points of the underlying Frobenius manifold, the sets of associated tautological relations coincide.*

[3] The proof of Theorem 3.3.6 easily implies that for a CohFT the only possible singularities of the R-matrix along D are poles.

[4] If the R-matrix were multivalued, we would need to work on a branched cover of U instead.

Remark 3.3.7. In the proof of Theorem 3.3.6 we will first locally near a smooth point on the discriminant identify canonical coordinates and normalized idempotents. An important part of the proof is to show that under this identification the quotient of corresponding R-matrices is holomorphic along the discriminant.

In [10] for the comparison of equivariant $\mathbb{P}^1$ and the A_2-singularity a different, more explicit identification of coordinates and vector fields is chosen. Therefore, while with this identification the quotient of the R-matrices is not holomorphic along the discriminant, there exists a holomorphic function φ such that $R_{\mathbb{P}^1}(z) = R(z)R_{A_2}(\varphi z)$. This result depends on the fact that the A_2-theory has an Euler vector field.

3.4 Local extension

The proof of the following theorem will occupy this section. The content of the proof is also used for proving Theorem 3.3.6.

Theorem 3.4.1. *Let M be an N-dimensional semisimple Frobenius manifold and let p be a smooth point of the discriminant of M such that $m = \frac{1}{2}$ in Theorem 2.3.10. Then, after possibly shrinking M to a smaller neighborhood of p, there exists a convergent CohFT with underlying Frobenius manifold M.*

We first study the consequences of Theorem 2.3.10 in more detail. After possibly shrinking M, it gives us a basis of holomorphic vector fields $\{\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u_{\geq 3}}\}$, where

$$\frac{\partial}{\partial t_0} = \frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}, \qquad \frac{\partial}{\partial t} = \left(\frac{3}{4}(u_1 - u_2)\right)^{\frac{1}{3}} \left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right). \tag{10}$$

It is easy to see that these vector fields commute and therefore we can integrate them to coordinates t_0, t and $u_{\geq 3}$. The discriminant D is then locally given by the equation $t = 0$.

Notice that there is a root $\sqrt{t}$ of t such that

$$\frac{\partial}{\partial t} = \sqrt{t}\left(\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}\right), \qquad \left(\frac{\partial}{\partial t}\right)^2 = t\left(\frac{\partial}{\partial u_1} + \frac{\partial}{\partial u_2}\right).$$

Define holomorphic functions η_0 and η_1 by

$$\eta_0 = \eta\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_0}\right), \qquad \eta_1 = \eta\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t_0}\right)$$

and notice that

$$\eta\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \eta\left(\frac{\partial}{\partial t} \star \frac{\partial}{\partial t}, \sum_{i=1}^{N} \frac{\partial}{\partial u_i}\right) = t\eta_0.$$

Since η is nonsingular, η_1 cannot vanish on the discriminant. The inverses Δ_1, Δ_2 of the norms of the first two idempotents are given by

$$\Delta_1 = \frac{2\sqrt{t}}{\eta_1 + \sqrt{t}\eta_0}, \qquad \Delta_2 = \frac{-2\sqrt{t}}{\eta_1 - \sqrt{t}\eta_0}.$$

We next choose roots $\sqrt{2\sqrt{t}}$, $\sqrt{-2\sqrt{t}}$ and $\sqrt{\eta_1}$. These induce roots of Δ_1, Δ_2, which we will use to define the normalized idempotents. Let Ψ_0 be the block diagonal matrix with upper left block being

$$\begin{pmatrix} \frac{\sqrt{t}}{\sqrt{2\sqrt{t}}} & \frac{-\sqrt{t}}{\sqrt{-2\sqrt{t}}} \\ \frac{1}{\sqrt{2\sqrt{t}}} & \frac{1}{\sqrt{-2\sqrt{t}}} \end{pmatrix} \tag{11}$$

and the identity matrix as the lower right block. For Ψ_0^{-1} the upper left block is given by

$$\begin{pmatrix} \frac{1}{\sqrt{2\sqrt{t}}} & \frac{\sqrt{t}}{\sqrt{2\sqrt{t}}} \\ \frac{1}{\sqrt{-2\sqrt{t}}} & \frac{-\sqrt{t}}{\sqrt{-2\sqrt{t}}} \end{pmatrix}. \tag{12}$$

The matrix Ψ_0 is the basis change from normalized idempotents to the basis $\{\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u_{\geq 3}}\}$. In the A_2-singularity case $\eta_0 = 0$, $\sqrt{\eta_1} = 1$ and $\sqrt{\Delta_{\geq 3}} = 1$.

Let Ψ_1 denote the basis change from the normalized idempotent basis to a flat basis and define $\tilde{\Psi}_1 = \Psi_1 \Psi_0^{-1}$.

Lemma 3.4.2. *The basis change matrix $\tilde{\Psi}_1$ is holomorphic along D.*

Proof. By Theorem 2.3.10 it is enough to prove the same statement for $\tilde{\Psi}' := \Psi' \Psi_0^{-1}$ where Ψ' is the basis change from the normalized idempotent basis to the basis $\{\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u_{\geq 3}}\}$. Since the basis changes leave all but the first two idempotents invariant we will only need to consider the upper-left 2×2 block of $\tilde{\Psi}'$. We factor this block into

$$\begin{pmatrix} \frac{\sqrt{t}}{2\sqrt{t}} & \frac{-\sqrt{t}}{-2\sqrt{t}} \\ \frac{1}{2\sqrt{t}} & \frac{1}{-2\sqrt{t}} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{\Delta_1}}{\sqrt{2t}} & 0 \\ 0 & \frac{\sqrt{\Delta_2}}{\sqrt{-2t}} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{t} \\ 1 & -\sqrt{t} \end{pmatrix},$$

a change from $\{\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t}\}$ to the idempotents, a multiplication by a diagonal matrix and the change back from the idempotents to $\{\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t}\}$. So we see that the upper left block of $\tilde{\Psi}'$ has the form

$$\eta_1^{-\frac{1}{2}} \begin{pmatrix} a & tc \\ c & a \end{pmatrix},$$

where

$$a = \frac{1}{2} \left(\frac{1}{\sqrt{1 + \sqrt{t}\frac{\eta_0}{\eta_1}}} + \frac{1}{\sqrt{1 - \sqrt{t}\frac{\eta_0}{\eta_1}}} \right) = 1 + \frac{3}{8}\frac{\eta_0^2}{\eta_1^2} t + O(t^2)$$

$$c = \frac{1}{2\sqrt{t}} \left(\frac{1}{\sqrt{1 + \sqrt{t}\frac{\eta_0}{\eta_1}}} - \frac{1}{\sqrt{1 - \sqrt{t}\frac{\eta_0}{\eta_1}}} \right) = -\frac{1}{2}\frac{\eta_0}{\eta_1} - \frac{5}{16}\frac{\eta_0^3}{\eta_1^3} t + O(t^2)$$

which are holomorphic along the discriminant. □

We can repeat this setup for any other N-dimensional Frobenius manifold satisfying the assumptions, in particular, as we will assume from now on, for any Frobenius manifold underlying an N-dimensional convergent

CohFT Ω^2, such as an extension of the theory of the A_2-singularity to N dimensions using the construction of Example 3.1.6 repeatedly. By using the coordinates $t, t_0, u_{\geq 3}$ we can identify a small neighborhood of p with a small neighborhood of the origin of the convergent CohFT. Let us shrink M accordingly. Notice that this isomorphism of complex manifolds, if the third roots in (10) have been chosen compatibly, amounts, outside of the discriminant, to identifying their canonical coordinates. We accordingly identify normalized idempotents and thereby the tangent spaces that they span.

This identification preserves the metric but not the quantum product structure. In particular, the basis change Ψ_2 from the normalized idempotent basis to a flat basis of the CohFT in general does not agree with Ψ_1. We set $\tilde{\Psi}_2 = \Psi_2\Psi_0^{-1}$, which by Lemma 3.4.2 is also holomorphic along D.

Lemma 3.4.3. *There exists a symplectic solution R_1 of the flatness equation (8) for Ψ_1 such that if R_2 denotes the solution of (8) for Ψ_2 used for reconstructing the CohFT, the endomorphism $R_1R_2^{-1}$ is holomorphic (under the identifications we have made above).*

Proof. For this proof let R_1 and R_2 denote R-matrices written in the basis of normalized idempotents instead of the underlying endomorphism valued power series. We set

$$R_i = \Psi_0^{-1}\tilde{R}_i\Psi_0,$$

where Ψ_0 is as in (11). We can write the flatness equations (8) as

$$[\tilde{R}_i, \Psi_0 \mathrm{d}u\Psi_0^{-1}] + z\tilde{\Psi}_i^{-1}\mathrm{d}(\tilde{\Psi}_i\tilde{R}_i\Psi_0)\Psi_0^{-1} = 0.$$

If $R := \tilde{R}_1\tilde{R}_2^{-1}$, these two differential equations combine to

$$0 = [R, \Psi_0\mathrm{d}u\Psi_0^{-1}] + z\tilde{\Psi}_1^{-1}\mathrm{d}(\tilde{\Psi}_1R\tilde{\Psi}_2^{-1})\tilde{\Psi}_2. \tag{13}$$

By Lemma 3.4.2 it is enough to show that there is a solution R of (13) all of whose entries are holomorphic along the discriminant and which satisfies the symplectic condition.

We analyze the entries of the ingredients in (13). For this we will consider all matrices to consist of four blocks numbered according to

$$\begin{pmatrix} 1. & 2. \\ 3. & 4. \end{pmatrix},$$

such that the first block has size 2×2. By Lemma 3.4.2 the matrices $\tilde{\Psi}_i^{-1}$, $\tilde{\Psi}_i$ for $i \in \{1, 2\}$ and therefore also the matrices $\tilde{\Psi}_1^{-1}\mathrm{d}\tilde{\Psi}_1$ and $(\mathrm{d}\tilde{\Psi}_2^{-1})\tilde{\Psi}_2$ of one-forms are holomorphic along D. The matrix $(\mathrm{d}\Psi_0)\Psi_0^{-1}$ has all blocks equal to zero except for the first one, which is

$$\frac{\mathrm{d}t}{4t}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{14}$$

and the matrix $\Psi_0\mathrm{d}\mathbf{u}\Psi_0^{-1}$ is block diagonal with first block being

$$\begin{pmatrix} \mathrm{d}t_0 & t\mathrm{d}t \\ \mathrm{d}t & \mathrm{d}t_0 \end{pmatrix} \tag{15}$$

and the other block being the diagonal matrix with entries $\mathrm{d}u_{\geq 3}$. Furthermore, because in general $\Psi_1^{-1}\mathrm{d}\Psi_1$ is antisymmetric (as can be seen by differentiating $\Psi_1^t\eta\Psi_1 = 1$) and from (11), (12) and (14) we see that the first block of $\tilde{\Psi}_1^{-1}\mathrm{d}\tilde{\Psi}_1$ has the general form

$$\begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \tag{16}$$

and the forth block is still antisymmetric. The same holds for $(\mathrm{d}\tilde{\Psi}_2^{-1})\tilde{\Psi}_2$.

We construct the coefficients of R inductively. Let us set

$$R(z) = \sum_{i=0}^{\infty} R^i z^i$$

and R^i_{jk} for the entries of R^i. We assume that we have already constructed R^j for $j \leq i$ satisfying the flatness equation and symplectic condition modulo z^{i+1}.

Because of (15), inserting $\frac{\partial}{\partial t_0}$ into the z^{i+1}-part of (13) directly gives equations for the off-diagonal blocks of R^{i+1} in terms of holomorphic functions.

Similarly, inserting $\frac{\partial}{\partial u_{\geq 3}}$ into the z^{i+1}-part of (13) gives us holomorphic formulas for the off-diagonal entries of R^{i+1} in the forth block. For the diagonal entries of this block we instead insert $\frac{\partial}{\partial t}$ into the z^{i+2}-part of (13) and because of the antisymmetry obtain that the first t-derivatives of the diagonal entries are holomorphic. We can integrate them locally and have an arbitrary choice of integration constants (ignoring the symplectic condition for now).

It remains the analysis of the first block. For this it is useful to compute the commutator

$$\left[\begin{pmatrix} R^{i+1}_{11} & R^{i+1}_{12} \\ R^{i+1}_{21} & R^{i+1}_{22} \end{pmatrix}, \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix}\right] = \begin{pmatrix} R^{i+1}_{12} - tR^{i+1}_{21} & t(R^{i+1}_{11} - R^{i+1}_{22}) \\ R^{i+1}_{22} - R^{i+1}_{11} & tR^{i+1}_{21} - R^{i+1}_{12}. \end{pmatrix}$$

So the insertion of $\frac{\partial}{\partial t}$ into the z^{i+1}-part of (13) gives holomorphic formulas for $R^{i+1}_{11} - R^{i+1}_{22}$ and $R^{i+1}_{12} - tR^{i+1}_{21}$. Because of (14) and (16) the $\frac{\partial}{\partial t}$ insertion into the z^{i+2}-part of (13) shows that the first t-derivative of $R^{i+1}_{11} + R^{i+1}_{22}$ is holomorphic and therefore this sum is holomorphic and we again have the choice of an integration constant. Similarly, we find that

$$2t\frac{\partial}{\partial t}R^{i+1}_{21} + R^{i+1}_{21}.$$

is holomorphic in D and therefore R^{i+1}_{21} is holomorphic up to a possible constant multiple of $t^{-\frac{1}{2}}$. Here, we have a unique choice of integration constant giving a holomorphic solution.

In general the symplectic condition does not constrain the integration constants of R^{i+1} when $i+1$ is odd [7]. On the other hand, it completely determines the integration constants of R^{i+1} when $i+1$ is even. It is clear that the solution determined by the symplectic condition is meromorphic and hence by the above analysis is also holomorphic. $\square$

Let R_1 and R_2 be as in the lemma. We define a new convergent CohFT Ω^3 by the R-matrix action $\Omega^3 = (R_1 R_2^{-1})\Omega^2$. We want to compare this CohFT to the CohFT Ω^1 obtained by the R-matrix action of R_1 on the trivial CohFT. Notice that Ω^1 is possibly not well-defined along the discriminant. The CohFTs Ω^3 and Ω^1 are very similar but the underlying trivial theories do not agree.

Recall the description (9) of the reconstruction using the basis of normalized idempotents. A local contribution at a vertex of color i for the reconstruction of Ω^3 is of the form

$$\Delta_{2i}^{\frac{2g-2+n}{2}} \sum_{k=0}^{\infty} \frac{\Delta_{2i}^{\frac{k}{2}}}{k!} \pi_* \left(\prod_{j=1}^{n} \alpha_j^i \prod_{j=1}^{k} \psi_j (\mathrm{Id}^i - (R_1^{-1}(\psi_j))^i) \mathbf{1}_{\Omega^2} \right),$$

where π forgets the last k markings and α_j^i are some formal series in ψ_j whose coefficients are holomorphic functions on the Frobenius manifold.

To circumvent convergence issues, let v be a formal flat vector field and let v_μ and v_i be the coordinates of v when written in a basis of flat coordinates or in the basis of normalized idempotents, respectively. We can further modify the CohFT by shifting along $v\psi$:

$$\Omega^4_{g,n}(\alpha_1, \ldots, \alpha_n) := \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Omega^3_{g,n+k}(\alpha_1, \ldots, \alpha_n, \psi v, \ldots, \psi v) \tag{17}$$

We obtain a well-defined convergent CohFT defined over the ring of power series in the v_μ. For Ω^4 the local contribution at a vertex of color i is

$$\Delta_{2i}^{\frac{2g-2+n}{2}} \sum_{k=0}^{\infty} \frac{\Delta_{2i}^{\frac{k}{2}}}{k!} \pi_* \left(\prod_{j=1}^{n} \alpha_j^i \prod_{j=1}^{k} \psi_j \left[\mathrm{Id}^i \mathbf{1}_{\Omega^2} - (R_1^{-1}(\psi_j))^i (\mathbf{1}_{\Omega^2} - v) \right] \right).$$

Recall that the dilaton equation implies that

$$\frac{1}{(1-a)^{2g-2+n}} = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \left(\prod_{j=1}^{k} a\psi_j \right),$$

where a is a formal variable, or equivalently

$$1 = \sum_{k=0}^{\infty} \frac{(1+b)^{2-2g-n-k}}{k!} \pi_* \left(\prod_{j=1}^{k} b\psi_j \right),$$

where $b = a/(1-a)$. We will apply this identity locally at every vertex. At a vertex of color i we use $-\sqrt{\Delta_{2i}} v_i$ for b. Then the local contribution at a vertex of color i is

$$\Delta_{3i}^{\frac{2g-2+n}{2}} \sum_{k=0}^{\infty} \frac{\Delta_{3i}^{\frac{k}{2}}}{k!} \pi_* \left(\prod_{j=1}^{n} \alpha_j^i \prod_{j=1}^{k} \psi_j \left[\mathrm{Id}^i - (R_1^{-1}(\psi_j))^i \right] (\mathbf{1}_{\Omega^2} - v) \right),$$

where

$$\Delta_{3i}^{-1/2} = \Delta_{2i}^{-1/2} - v_i.$$

Notice that now (again) the sum in k is finite in each cohomological degree. Therefore we can specialize v. We will take v to the vector $\mathbf{1}_{\Omega^2} - \mathbf{1}_{\Omega^1}$, which is holomorphic by Lemma 3.4.2, and thus $v_i = \Delta_{2i}^{-1/2} - \Delta_{1i}^{-1/2}$. In this case the Δ_{3i} specialize to Δ_{1i}. We have therefore arrived exactly at the reconstruction formula for Ω^1. In particular, with the specialization of v, Ω^4 is the same as Ω^1 and therefore Ω^1 is also holomorphic along the discriminant. Hence Ω^1 is a suitable local extension of the Frobenius manifold we started with to a convergent CohFT. The extension in not unique but depends on a choice of integration constants.

3.5 Equivalence of relations

We want to prove Theorem 3.3.6 in this section.

First notice that the dimension of a convergent CohFT Ω can be increased by one without changing the set of relations by the construction of Example 3.1.6. So we can assume that the CohFTs we are trying to compare have the same dimension.

Next, recall from Section 3.3 that the tautological relations of a semisimple convergent CohFT Ω are defined via coefficients of the part of the Givental-Teleman classification singular in the discriminant. Therefore the relations do not change when removing the codimension two set of singular points of the discriminant from the Frobenius manifold underlying Ω.

In order to prove Theorem 3.3.6, it is therefore enough to show that the relations coincide for two semisimple, equal dimensional convergent CohFTs Ω^1, Ω^2 such that each Frobenius manifold contains a smooth point of the discriminant and is small enough for Theorem 3.4.1 to apply directly to Ω^1.

By the proof of Theorem 3.4.1 an extension of the Frobenius manifold underlying Ω^1 can be constructed from Ω^2 by a holomorphic R-matrix and a holomorphic shift. To prove Theorem 3.3.6 it therefore suffices to show that these two operations preserve tautological relations and that the integration constants can be chosen such that the constructed CohFT coincides with Ω^1. We now prove these statements.

Lemma 3.5.1. *The R-matrix action by a holomorphic R-matrix preserves tautological relations.*

Proof. Let Ω' be obtained from Ω from the R-matrix action of R. Then in the description of the R-matrix action in Section 3.2 all arguments are holomorphic vector fields on the Frobenius manifold with values in power series in ψ-classes. $\Omega'_{g,n}$ in each cohomological degree is obtained by a finite sum of push-forwards under the gluing map of products of Ω (with possibly additional markings) multiplied by monomials in ψ classes and with holomorphic vector fields as arguments. Therefore any singularities of the reconstruction of Ω' are the result of singularities in the reconstruction of Ω. So we can write the relations of Ω' from vanishing singularities in terms of the general relations from Ω as of Definition 3.3.1. By the stability condition in Definition 3.3.1 we can also express a general relation of Ω' in terms of relations from Ω.

Since R-matrices are power series starting with the identity matrix, by using R^{-1} we can also write the relations of Ω in terms of relations from Ω'. ∎

The shift-construction (17) clearly expresses any relation from $\Omega^4_{g,n}$ in terms of relations from $\Omega^3_{g,n+m}$ for various $m \geq 0$.

We now finally want to show that taking the R-matrices of Ω^1 and Ω^2 is a suitable choice for R_1 and R_2 in Lemma 3.4.3. We will argue that otherwise Ω^1 or Ω^2 will not be defined at the discriminant.

For simplicity we will make use of the following stability result. It should also be possible to use estimates or congruence properties of intersection numbers instead.

Theorem 3.5.2 (Boldsen [1], Looijenga [12])**.** *For $k < \frac{g}{3}$ the vector space $H^{2k}(M_{g,n})$ is freely generated by the set of monomials in the classes $\kappa_1, \ldots, \kappa_k, \psi_1, \ldots, \psi_n$ of cohomological degree $2k$.*

We use the local coordinates $t, t_0, u_{\geq 3}$ from the previous section. Let i be the lowest degree in z where R is not holomorphic. The non-holomorphic part is a constant multiple of the block-diagonal matrix with upper-left block

$$\begin{pmatrix} 0 & t^{1/2} \\ t^{-1/2} & 0 \end{pmatrix}$$

and zeros everywhere else. Let us consider the ψ_1^i-coefficient of

$$\Omega^1_{g,1}\left(\frac{\partial}{\partial t_0}\right)\Big|_{M_{g,1}} - \Omega^2_{g,1}\left(\frac{\partial}{\partial t_0}\right)\Big|_{M_{g,1}}$$

for large g. Its lowest order term in t is up to nonzero factors given by

$$t^{-\frac{1}{2}}\sqrt{t}\left(\sqrt{2\sqrt{t}}^{\,2g-2+1-1} - \sqrt{-2\sqrt{t}}^{\,2g-2+1-1}\right)$$
$$= 2^{g-1}\left((\sqrt{t})^{g-1} - (-\sqrt{t})^{g-1}\right)$$

and therefore not holomorphic in t for even g. By Theorem 3.5.2 this is impossible.

3.6 Global extension

The local extension Theorem 3.4.1 leaves open the question when a semisimple Frobenius manifold can (globally) be extended to a CohFT. In Section 3.7.2 we will see that the restrictions put on integration constants of the R-matrices in Lemma 3.4.3 do not always fit together globally.

Conjecture 3.6.1. *Let M be an N-dimensional semisimple Frobenius manifold such that it possesses a holomorphic genus one potential* $\mathrm{d}G$*. Then there exists a convergent CohFT with underlying Frobenius manifold M.*

On the other hand, when the Frobenius manifold is homogeneous such an extension to a CohFT exists by the following simple argument. There is a unique homogeneous solution to the flatness equation (8) and by construction it is meromorphic along the discriminant. Since by Lemma 3.4.3 all possible solutions are either holomorphic in the discriminant or are multivalued, the homogeneous solution has in fact to be holomorphic.

3.7 Examples

3.7.1 Extending the comparison to non-smooth points on the discriminant

We want to illustrate how the comparison between relations in the proof of Theorem 3.3.6 via an identification of coordinates and vector fields, an R-matrix action and a shift, does not directly extend to give a way to explicitly write the relations near a singular point of the discriminant in terms of the A_2- (3-spin) relations.

Let us consider the comparison between the $A_2 \times A_1$ and A_3 singularities. We will see that already the identification between points and vector fields behaves badly. This is the simplest example we can consider since in two dimensions the discriminant locus is a union of parallel lines and in particular is non-singular.

The Frobenius manifold of the A_3-singularity $x^4/4 = 0$ is based on the versal deformation space

$$f(x) = \frac{x^4}{4} + t_2 x^2 + t_1 x + t_0.$$

Here t_0, t_1 and t_2 are coordinates on the Frobenius manifold. The ring structure is given by the Milnor ring

$$\mathbb{C}[t_0, t_1, t_2][x]/f'(x),$$

where $x = \frac{\partial}{\partial t_1}$. The discriminant of the minimal polynomial f' of x is $-32t_2^3 - 27t_1^2$ and therefore the discriminant locus has a cusp at $t_1 = t_2 = 0$. The metric is in the basis $\{1, x, x^2\}$ given by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2t_2 \end{pmatrix}.$$

Therefore the basis $\{1, x, x^2\}$ is flat up to a determinant one basis change.

We go to a sixfold ramified cover of the Frobenius manifold on which we can define the critical points $\zeta_1, \zeta_2, \zeta_3$ of $f(x)$ as holomorphic functions. Let u_1, u_2, u_3 be the corresponding critical values. Part of the discriminant locus is described by the equation $\zeta_1 = \zeta_2$. Locally we use $\phi := \zeta_1 - \zeta_2$, ζ_3 and t_0 as new coordinates. Reexpressing in terms of the coordinates gives

$$\begin{aligned}
\zeta_1 &= -\frac{1}{2}\zeta_3 + \frac{1}{2}\phi, \\
\zeta_2 &= -\frac{1}{2}\zeta_3 - \frac{1}{2}\phi, \\
u_1 &= t_0 + \frac{3}{64}\zeta_3^4 - \frac{5}{32}\zeta_3^2\phi^2 + \frac{1}{8}\zeta_3\phi^3 - \frac{1}{64}\phi^4, \\
u_2 &= t_0 + \frac{3}{64}\zeta_3^4 - \frac{5}{32}\zeta_3^2\phi^2 - \frac{1}{8}\zeta_3\phi^3 - \frac{1}{64}\phi^4, \\
u_1 - u_2 &= \frac{1}{4}\zeta_3\phi^3, \\
u_3 &= t_0 - \frac{3}{8}\zeta_3^4 + \frac{1}{8}\zeta_3^2\phi^2.
\end{aligned}$$

The idempotents are given by

$$\begin{aligned}
\frac{\partial}{\partial u_1} &= \frac{(x-\zeta_2)(x-\zeta_3)}{(\zeta_1-\zeta_2)(\zeta_1-\zeta_3)} = \frac{x^2+(-\frac{1}{2}\zeta_3+\frac{1}{2}\phi)x-\frac{1}{2}\zeta_3^2-\frac{1}{2}\zeta_3\phi}{-\frac{3}{2}\zeta_3\phi+\frac{1}{2}\phi^2},\\
\frac{\partial}{\partial u_2} &= \frac{(x-\zeta_1)(x-\zeta_3)}{(\zeta_2-\zeta_1)(\zeta_2-\zeta_3)} = \frac{x^2+(-\frac{1}{2}\zeta_3-\frac{1}{2}\phi)x-\frac{1}{2}\zeta_3^2+\frac{1}{2}\zeta_3\phi}{\frac{3}{2}\zeta_3\phi+\frac{1}{2}\phi^2},\\
\frac{\partial}{\partial u_3} &= \frac{(x-\zeta_1)(x-\zeta_2)}{(\zeta_3-\zeta_1)(\zeta_3-\zeta_2)} = \frac{x^2+\zeta_3x+\frac{1}{4}\zeta_3^2-\frac{1}{4}\phi^2}{\frac{9}{4}\zeta_3^2-\frac{1}{4}\phi^2}
\end{aligned}$$

so that they become after normalization

$$\frac{x^2+(-\frac{1}{2}\zeta_3+\frac{1}{2}\phi)x-\frac{1}{2}\zeta_3^2-\frac{1}{2}\zeta_3\phi}{\sqrt{-\frac{3}{2}\zeta_3\phi+\frac{1}{2}\phi^2}}, \frac{x^2+(-\frac{1}{2}\zeta_3-\frac{1}{2}\phi)x-\frac{1}{2}\zeta_3^2+\frac{1}{2}\zeta_3\phi}{\sqrt{\frac{3}{2}\zeta_3\phi+\frac{1}{2}\phi^2}}, \frac{x^2+\zeta_3x+\frac{1}{4}\zeta_3^2-\frac{1}{4}\phi^2}{\sqrt{\frac{9}{4}\zeta_3^2-\frac{1}{4}\phi^2}}.$$

For $A_2 \times A_1$, let us assume that u_3 corresponds to the A_1-direction and that the norm of the idempotent in that direction is one. We can write

$$u_1 = x_0 - \frac{2}{3}(-x_1)^{3/2}, \quad u_2 = x_0 + \frac{2}{3}(-x_1)^{3/2},$$

where x_0 and x_1 are flat coordinates corresponding to t_0 and t_1 in Example 2.2.2.

We should therefore identify

$$\phi \stackrel{!}{=} -2\left(\frac{2}{3}\right)^{1/3} \zeta_3^{-1/3}\sqrt{-x_1}.$$

Let us consider how we identify the A_3-singularity basis $\{1, x, x^2\}$ and the flat basis of $A_2 \times A_1$ via the identification of their normalized idempotents. If we write the identity of $A_2 \times A_1$ in terms of $\{1, x, x^2\}$, the x^2-coefficient is

$$\begin{aligned}
&\frac{\sqrt{2\sqrt{-x_1}}}{\sqrt{-\frac{3}{2}\zeta_3\phi+\frac{1}{2}\phi^2}} + \frac{\sqrt{-2\sqrt{-x_1}}}{\sqrt{\frac{3}{2}\zeta_3\phi+\frac{1}{2}\phi^2}} + \frac{1}{\sqrt{\frac{9}{4}\zeta_3^2-\frac{1}{4}\phi^2}}\\
&\qquad = \frac{2\zeta_3^{1/6}}{\sqrt{-3c\zeta_3+c\phi}} + \frac{2\zeta_3^{1/6}}{\sqrt{3c\zeta_3+c\phi}} + \frac{2}{\sqrt{9\zeta_3^2-\phi^2}},
\end{aligned}$$

where

$$c = -2\left(\frac{2}{3}\right)^{1/3}.$$

The coefficient is well-defined on generic points of the part $\zeta_1 = \zeta_2$ ($\phi = 0$) of the discriminant, but when fixing some $\phi \neq 0$ the function has a singularity at $\zeta_3 = 0$.

3.7.2 Obstructions to extending R-matrices

We want consider the class of two-dimensional Frobenius manifolds with flat coordinates t_0, t, flat metric

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and quantum product

$$\left(\frac{\partial}{\partial t}\right)^2 = f \frac{\partial}{\partial t_0}$$

for a holomorphic function $f(t)$. The corresponding Gromov-Witten potential is

$$\frac{1}{2} t_0^2 t + F,$$

where $F(t)$ is a third anti-derivative of $f(t)$.

The differential equation satisfied by the R-matrix in flat coordinates can be made explicit:

$$\left[R, \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix}\right] + z\dot{R} + z\frac{\dot{f}}{4f}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} R = 0 \tag{18}$$

We first want to show that for any solution R, the z^1-coefficient is not holomorphic for all f. For this we set

$$R = \begin{pmatrix} 1 + az & 0 + bz \\ 0 + cz & 1 + dz \end{pmatrix} + O(z^2).$$

From (18) in degree z^1 we obtain

$$b - fc - \frac{\dot{f}}{4f} = 0, \qquad a = d.$$

From (18) in degree z^2, we see that $a = d$ is an integration constant. We obtain an interesting differential equation for c:

$$2f\dot{c} + \dot{f}c + \frac{\ddot{f}}{4f} - \frac{5\dot{f}^2}{16f^2} = 0$$

If we substitute

$$c = \frac{\gamma}{f} - \frac{5}{48}\frac{\dot{f}}{f^2},$$

it becomes

$$2\dot{\gamma} - \frac{\dot{f}}{f}\gamma + \frac{\ddot{f}}{24f} = 0.$$

So γ is determined up to a multiple of a root of f and in particular, if f has somewhere a simple zero, there exists at most one solution meromorphic on all of $\mathbb{C}^2$.

If f is linear, $\gamma = 0$ is clearly a holomorphic solution. If f is quadratic with non-vanishing discriminant, there is still a holomorphic solution. For example for

$$f(t) = t(t+1)$$

the solution is

$$\gamma = \frac{t}{6} + \frac{1}{12}.$$

In larger degree, we stop having meromorphic solutions. In the example

$$f(t) = t(t^2 - 1),$$

after substituting

$$\gamma = f\delta + \frac{t}{8},$$

we arrive at the differential equation

$$(t^2 - 1)2t\dot{\delta} + (3t^2 - 1)\delta + \frac{1}{8} = 0.$$

We see that δ is meromorphic in t if and only it is so in $u := t^2$. In the new variable the differential equation is

$$4u(u-1)\delta' + (3u-1)\delta + \frac{1}{8} = 0.$$

From generic semisimplicity we also know that δ has to be holomorphic except for $u = 0$ and $u = 1$. Around $u = 0$ and $u = 1$ there are unique meromorphic solutions

$$\frac{1}{8}\sum_{i=0}^{\infty} \frac{4i+3}{4i+1} u^i, \qquad -\frac{1}{16}\sum_{i=0}^{\infty} \frac{4i+3}{4i+2}(1-u)^i,$$

but these obviously do not agree.

We now want to check that the corresponding genus one potential will also be singular. For this we look at the case when $a = d = 0$ and compute the codimension one part of the reconstructed CohFT on $\overline{M}_{1,1}$ with an $\frac{\partial}{\partial t}$-insertion. From the trivial graph, we obtain the contribution

$$-2\left(\gamma + \frac{7}{48}\frac{\dot{f}}{f}\right)\psi_1 + 2\left(\gamma - \frac{5}{48}\frac{\dot{f}}{f}\right)\kappa_1$$

and in addition we have the contribution

$$2\gamma + \frac{2}{48}\frac{\dot{f}}{f}$$

of the irreducible divisor δ_0. From

$$\int_{\overline{M}_{1,1}} \psi_1 = \int_{\overline{M}_{1,1}} \kappa_1 = \frac{1}{12}\int_{\overline{M}_{1,1}} \delta_0 = \frac{1}{24}$$

we see that the correlator equals γ, which is not holomorphic on all of the Frobenius manifold.

3.8 Other relations from cohomological field theories

For a convergent CohFT depending on additional parameters there are possibilities to obtain tautological relations from the reconstruction, which are different from Definition 3.3.1. We want to study here the example of the equivariant Gromov-Witten theory of a toric variety, which is dependent on equivariant and Novikov parameters.

Let $T = (\mathbb{C}^*)^m$ and let $H_T^*(\mathrm{pt}) = H^*(BT) = \mathbb{C}[\lambda_1, \dots, \lambda_m]$ be the T-equivariant cohomology ring of a point. Let X be an m-dimensional smooth, toric variety with a basis $\{p_1, \dots, p_N\}$ of its cohomology, which we can also lift it to a basis in T-equivariant cohomology. Let $\beta_1, \dots, \beta_N$ be the dual basis in homology. The Novikov ring is a completion of the semigroup ring of effective classes $\beta \in H_2(X; \mathbb{Z})$. We use q^β to denote the generator corresponding to a $\beta \in H_2(X; \mathbb{Z})$.

A family of N-dimensional CohFTs on the state space $H^*_{\mathbb{C}^*}(X)$ can be defined by setting

$$\Omega_{g,n}(\alpha_1, \dots, \alpha_n) = \sum_\beta q^\beta p_* \left(\prod_{i=1}^n \mathrm{ev}_i^*(\alpha_i) \cap [\overline{M}_{g,n}(X; \beta)]^{vir} \right),$$

where the sum ranges over all effective classes $\beta \in H_2(X; \mathbb{Z})$, p is the projection from the moduli space of stable maps to $\overline{M}_{g,n}$ and ev_i is the ith evaluation map. From [9] it follows that the sum over β converges in a neighborhood of the origin and that the CohFT induces a convergent CohFT. Iritani also shows that this convergent CohFT is semisimple. We can view its classes as holomorphically varying in the parameters $\lambda_1, \dots, \lambda_m$ and parameters $q_1, \dots, q_{N_1}$ corresponding to a nef basis of $H_2(X; \mathbb{Z})$. There are flat coordinates $t_1, \dots, t_{N_1}$ of the Frobenius manifold corresponding to the Poincaré dual basis of $H^2(X)$.

For any choice of $\lambda_1, \dots, \lambda_m, q_1, \dots, q_{N_1}$ such that the discriminant does not vanish identically, as before, we can define relations by studying the behavior of the Givental-Teleman reconstruction near the discriminant, and we know from Theorem 3.3.6 that these follow from Pixton's relations. Now we can however also allow to let the parameters λ_i, q_i vary and there might be additional pole cancellation in the reconstruction formula.[5] For example there might be terms having poles in the equivariant parameters. We want to show now that these relations also follow from Pixton's relations.

We consider a function f on the space of equivariant and torus parameters times the Frobenius manifold with values in the strata algebra $\mathcal{S}_{g,n}$ which is obtained from the reconstruction. We need to show that the projection $\bar{f}$ of f to the space of functions with values in $\mathcal{S}_{g,n}/\mathcal{P}_{g,n}$, the strata algebra divided the ideal of the relations of Pixton, becomes holomorphic. By Theorem 3.3.6 we know that $\bar{f}$ is holomorphic for any fixed values of λ_i and q_i such that the discriminant is not identically zero on the Frobenius manifold. We can conclude if we can show that the set

[5]To make this statement precise one needs to know that the R-matrix depends on the parameters λ_i, q_i also meromorphically. This follows from mirror symmetry [9].

of such bad λ_i, q_i in the space of all parameters is of codimension at least 2.

From the divisor equation it follows that all structure constants and therefore also the discriminant depend on q_i and the coordinate t_i only in the combination $q_i e^{t_i}$. Therefore the locus of bad parameters is a product $L \times Q$ (intersected with the domain of convergence), where $L \subseteq \mathbb{C}^m$ and $Q \subseteq \mathbb{C}^{N_1}$ correspond to the λ_i and q_i respectively, and where Q is a product of N_1 factors which are either $\{0\}$ or all of $\mathbb{C}$. If at least two factors of Q are $\{0\}$, the bad locus is of codimension at least 2. If there is exactly one factor of $\{0\}$, since the equivariant cohomology, which we obtain by setting all q_i to zero, is semisimple, L is of codimension at least one and we are also done in this case. Finally the case that Q has no factor $\{0\}$ means that the theory is not semisimple for any choice of q_i which clearly contradicts semisimplicity of the non-equivariant theory.

Remark 3.8.1. A similar strategy should also work for toric orbifolds. The special case of $\mathbb{P}^1$ with two orbifold points is used in [2].

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Curriculum Vitae

Felix Janda was born on 27 November 1990 in Munich, Germany. He has so far spent most of his life in the suburb Höhenkirchen, commuting to Munich for secondary school and his undergraduate studies at the Ludwigs-Maximilians Universität München. As a winner of the “Bundeswettbewerb Mathematik 2008” he was supported by the “Studienstiftung des deutschen Volkes” during his undergraduate studies. In 2011 he finished his diploma thesis under the direction of Ulrich Derenthal.

In the October of 2011 Felix has started his doctoral studies under the supervision of Rahul Pandharipande at ETH Zürich. Research done between 2012 and 2015 finally resulted in this thesis. During the time he has also served as a teaching assistant. In 2015, Felix has received a postdoctoral position of the Fondation Sciences Mathématiques de Paris and is looking forward to working with Dimtri Zvonkine at the Institut de Mathématiques de Jussieu in Paris starting from October 2015.